FUNDAMENTALS OF ORDINARY DIFFERENTIAL EQUATIONS — THE LECTURE NOTES FOR COURSE (189) 261/325

FUNDAMENTALS OF ORDINARY DIFFERENTIAL EQUATIONS

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Contents

Chapter 1

INTRODUCTION

1. Definitions and Basic Concepts

1.1 Ordinary Differential Equation (ODE)

An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in (I)$.

1.2 Solution

Any function $y = f(x)$ which satisfies this equation over the interval (I) is called a solution of the ODE.

For example, $y = e^{2x}$ is a solution of the ODE

$$y' = 2y$$

and $y = \sin(x^2)$ is a solution of the ODE

$$xy'' - y' + 4x^3 y = 0.$$

1.3 Order n of the DE

An ODE is said to be order n, if $y^{(n)}$ is the highest order derivative occurring in the equation. The simplest first order ODE is $y' = g(x)$.

The most general form of an n-th order ODE is

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

with F a function of $n+2$ variables $x, u_0, u_1, \ldots, u_n$. The equations

$$xy'' + y = x^3, \quad y' + y^2 = 0, \quad y''' + 2y' + y = 0$$

are examples of ODE's of second order, first order and third order respectively with respectively

$$\begin{aligned}&F(x, u_0, u_1, u_2) = xu_2 + u_0 - x^3,\\ &F(x, u_0, u_1) = u_1 + u_0^2,\\ &F(x, u_0, u_1, u_2, u_3) = u_3 + 2u_1 + u_0.\end{aligned}$$

1.4 Linear Equation:

If the function F is linear in the variables $u_0, u_1, \ldots, u_n$ the ODE is said to be **linear**. If, in addition, F is homogeneous then the ODE is said to be homogeneous. The first of the above examples above is linear are linear, the second is non-linear and the third is linear and homogeneous. The general n-th order linear ODE can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x).$$

1.5 Homogeneous Linear Equation:

The linear DE is homogeneous, if and only if $b(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, \ldots, y_m$ are solutions and $c_1, c_2, \ldots, c_m$ are constants then

$$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$$

is also a solution.

1.6 Partial Differential Equation (PDE)

An equation involving the partial derivatives of a function of more than one variable is called PED. The concepts of linearity and homogeneity can be extended to PDE's. The general second order linear PDE in two variables x, y is

$$\begin{aligned}a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x}\\ + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u = g(x,y).\end{aligned}$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a linear, homogeneous PDE of order 2. The functions $u = \log(x^2 + y^2)$, $u = xy$, $u = x^2 - y^2$ are examples of solutions of Laplace's equation. We will not study PDE's systematically in this course.

1.7 General Solution of a Linear Differential Equation

It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I). For example, the general solution of the differential equation $y' = 3x^2$ is $y = x^3 + C$ where C is an arbitrary constant. The constant C is the value of y at $x = 0$. This **initial condition** completely determines the solution. More generally, one easily shows that given a, b there is a unique solution y of the differential equation with $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^2 + C$ do not intersect one another and they fill up the plane $\mathcal{R}^2$.

1.8 A System of ODE's

$$\begin{aligned} y_1' &= G_1(x, y_1, y_2, \ldots, y_n) \\ y_2' &= G_2(x, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= G_n(x, y_1, y_2, \ldots, y_n) \end{aligned}$$

An n-th order ODE of the form $y^{(n)} = G(x, y, y', \ldots, y^{n-1})$ can be transformed in the form of the system of first order DE's. If we introduce dependant variables $y_1 = y, y_2 = y', \ldots, y_n = y^{n-1}$ we obtain the equivalent system of first order equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_n' &= G(x, y_1, y_2, \ldots, y_n). \end{aligned} \tag{1.1}$$

For example, the ODE $y'' = y$ is equivalent to the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_1. \end{aligned} \tag{1.2}$$

In this way the study of n-th order equations can be reduced to the study of systems of first order equations. Some times, one called the latter as the **normal form** of the n-th order ODE. Systems of equations arise in the study of the motion of particles. For example, if $P(x, y)$ is the position of a particle of mass m at time t, moving in a plane under the action of the force field $(f(x, y), g(x, y)$, we have

$$\begin{aligned} m\frac{d^2x}{dt^2} &= f(x, y), \\ m\frac{d^2y}{dt^2} &= g(x, y). \end{aligned} \tag{1.3}$$

This is a second order system.

The general first order ODE in normal form is

$$y' = F(x, y).$$

If F and $\frac{\partial F}{\partial y}$ are continuous one can show that, given a, b, there is a unique solution with $y(a) = b$. Describing this solution is not an easy task and there are a variety of ways to do this. The dependence of the solution on initial conditions is also an important question as the initial values may be only known approximately.

The non-linear ODE $yy' = 4x$ is not in normal form but can be brought to normal form

$$y' = \frac{4x}{y}.$$

by dividing both sides by y.

2. The Approaches of Finding Solutions of ODE

2.1 Analytical Approaches

- Analytical solution methods: finding the exact form of solutions;
- Geometrical methods: finding the qualitative behavior of solutions;
- Asymptotic methods: finding the asymptotic form of the solution, which gives good approximation of the exact solution.

2.2 Numerical Approaches

- Numerical algorithms — numerical methods;
- Symbolic manipulators — Maple, MATHEMATICA, MacSyma.

This course mainly discuss the analytical approaches and mainly on analytical solution methods.

Chapter 2

FIRST ORDER DIFFERENTIAL EQUATIONS

PART (I): LINEAR EQUATIONS

In this lecture we will treat linear and separable first order ODE's.

1. Linear Equation

The general first order ODE has the form $F(x, y, y') = 0$ where $y = y(x)$. If it is linear it can be written in the form

$$a_0(x)y' + a_1(x)y = b(x)$$

where $a_0(x),\ a_(x),\ b(x)$ are continuous functions of x on some interval (I). To bring it to normal form $y' = f(x, y)$ we have to divide both sides of the equation by $a_0(x)$. This is possible only for those x where $a_0(x) \neq 0$. After possibly shrinking I we assume that $a_0(x) \neq 0$ on (I). So our equation has the form (standard form)

$$y' + p(x)y = q(x)$$

with $p(x) = a_1(x)/a_0(x)$ and $q(x) = b(x)/a_0(x)$, both continuous on (I). Solving for y' we get the normal form for a linear first order ODE, namely

$$y' = q(x) - p(x)y.$$

1.1 Linear homogeneous equation

Let us first consider the simple case: $q(x) = 0$, namely,

$$\frac{\mathrm{d}y}{\mathrm{dx}} + p(x)y = 0.$$

With the chain law of derivative, one may write

$$\frac{y'(x)}{y} = \frac{\mathrm{d}}{\mathrm{d}x} \ln\left[y(x)\right] = -p(x),$$

integrating both sides, we derive

$$\ln y(x) = -\int p(x)\mathrm{d}x + C,$$

or

$$y = C_1 \mathrm{e}^{-\int p(x)\mathrm{d}x},$$

where C, as well as $C_1 = \mathrm{e}^C$, is arbitrary constant.

1.2 Linear inhomogeneous equation

We now consider the general case:

$$\frac{\mathrm{d}y}{\mathrm{dx}} + p(x)y = q(x).$$

We multiply the both sides of our differential equation with a factor $\mu(x) \neq 0$. Then our equation is equivalent (has the same solutions) to the equation

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x).$$

We wish that with a properly chosen function $\mu(x)$,

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y(x)].$$

For this purpose, the function $\mu(x)$ must has the property

$$\mu'(x) = p(x)\mu(x), \tag{2.1}$$

and $\mu(x) \neq 0$ for all x. By solving the linear homogeneous equation (2.1), one obtain

$$\mu(x) = e^{\int p(x)\mathrm{d}x}. \tag{2.2}$$

With this function, which is called an integrating factor, our equation is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y(x)] = \mu(x)q(x), \tag{2.3}$$

Integrating both sides, we get

$$\mu(x)y = \int \mu(x)q(x)\mathrm{d}x + C$$

with C an arbitrary constant. Solving for y, we get

$$y = \frac{1}{\mu(x)}\int \mu(x)q(x)\mathrm{d}x + \frac{C}{\mu(x)} = y_P(x) + y_H(x) \tag{2.4}$$

as the general solution for the general linear first order ODE

$$y' + p(x)y = q(x).$$

In solution (2.4), the first part, $y_P(x)$, is a particular solution of the inhomogeneous equation, while the second part, $y_H(x)$, is the general solution of the associate homogeneous solution. Note that for any pair of scalars a, b with a in (I), there is a unique scalar C such that $y(a) = b$. Geometrically, this means that the solution curves $y = \phi(x)$ are a family of non-intersecting curves which fill the region $I \times \mathcal{R}$.

Example 1: $y' + xy = x$. This is a linear first order ODE in standard form with $p(x) = q(x) = x$. The integrating factor is

$$\mu(x) = e^{\int x\mathrm{d}x} = e^{x^2/2}.$$

Hence, after multiplying both sides of our differential equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{x^2/2}y) = xe^{x^2/2}$$

which, after integrating both sides, yields

$$e^{x^2/2}y = \int xe^{x^2/2}\mathrm{d}x + C = e^{x^2/2} + C.$$

Hence the general solution is $y = 1 + Ce^{-x^2/2}$. The solution satisfying the initial condition $y(0) = 1$ is $y = 1$ and the solution satisfying $y(0) = a$ is $y = 1 + (a-1)e^{-x^2/2}$.

Example 2: $xy' - 2y = x^3 \sin x$,
$(x > 0)$. We bring this linear first order equation to standard form by dividing by x. We get

$$y' + \frac{-2}{x}y = x^2 \sin x.$$

The integrating factor is

$$\mu(x) = e^{\int -2\mathrm{d}x/x} = e^{-2\ln x} = 1/x^2.$$

After multiplying our DE in standard form by $1/x^2$ and simplifying, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(y/x^2) = \sin x$$

from which $y/x^2 = -\cos x + C$ and $y = -x^2 \cos x + Cx^2$. Note that the later are solutions to the DE $xy' - 2y = x^3 \sin x$ and that they all satisfy the initial condition $y(0) = 0$. This non-uniqueness is due to the fact that $x = 0$ is a singular point of the DE.

PART (II): SEPARABLE EQUATIONS — NONLINEAR EQUATIONS (1)

2. Separable Equations.

The first order ODE $y' = f(x, y)$ is said to be separable if $f(x, y)$ can be expressed as a product of a function of x times a function of y. The DE then has the form $y' = g(x)h(y)$ and, dividing both sides by $h(y)$, it becomes

$$\frac{y'}{h(y)} = g(x).$$

Of course this is not valid for those solutions $y = y(x)$ at the points where $\phi(x) = 0$. Assuming the continuity of g and h, we can integrate both sides of the equation to get

$$\int \frac{y'(x)}{h[y(x)]}\mathrm{d}x = \int g(x)\mathrm{d}x + C.$$

Assume that

$$H(y) = \int \frac{\mathrm{d}y}{h(y)},$$

By chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}H[y(x)] = H'(y)y'(x) = \frac{1}{h[y(x)]}y'(x),$$

hence

$$H[y(x)] = \int \frac{y'(x)}{h[y(x)]}\mathrm{d}x = \int g(x)\mathrm{d}x + C.$$

Therefore,

$$\int \frac{\mathrm{d}y}{h(y)} = H(y) = \int g(x)\mathrm{d}x + C,$$

gives the implicit form of the solution. It determines the value of y implicitly in terms of x.

Example 1: $y' = \frac{x-5}{y^2}$.

To solve it using the above method we multiply both sides of the equation by y^2 to get

$$y^2 y' = (x-5).$$

Integrating both sides we get $y^3/3 = x^2/2 - 5x + C$. Hence,

$$y = \left[3x^2/2 - 15x + C_1\right]^{1/3}.$$

Example 2: $y' = \frac{y-1}{x+3}$ $(x > -3)$. By inspection, $y = 1$ is a solution. Dividing both sides of the given DE by $y-1$ we get

$$\frac{y'}{y-1} = \frac{1}{x+3}.$$

This will be possible for those x where $y(x) \neq 1$. Integrating both sides we get

$$\int \frac{y'}{y-1}\mathrm{d}x = \int \frac{\mathrm{d}x}{x+3} + C_1,$$

from which we get $\ln|y-1| = \ln(x+3) + C_1$. Thus $|y-1| = e^{C_1}(x+3)$ from which $y - 1 = \pm e^{C_1}(x+3)$. If we let $C = \pm e^{C_1}$, we get

$$y = 1 + C(x+3)$$

which is a family of lines passing through $(-3, 1)$; for any (a, b) with $b \neq 0$ there is only one member of this family which passes through (a, b). Since $y = 1$ was found to be a solution by inspection the general solution is

$$y = 1 + C(x+3),$$

where C can be any scalar.

Example 3: $y' = \frac{y\cos x}{1+2y^2}$. Transforming in the standard form then integrating both sides we get

$$\int \frac{(1+2y^2)}{y}\mathrm{d}y = \int \cos x \,\mathrm{d}x + C,$$

from which we get a family of the solutions:

$$\ln|y| + y^2 = \sin x + C,$$

where C is an arbitrary constant. However, this is not the general solution of the equation, as it does not contains, for instance, the solution: $y = 0$. With I.C.: y(0)=1, we get $C = 1$, hence, the solution:

$$\ln|y| + y^2 = \sin x + 1.$$

3. Logistic Equation

$$y' = ay(b - y),$$

where $a, b > 0$ are fixed constants. This equation arises in the study of the growth of certain populations. Since the right-hand side of the equation is zero for $y = 0$ and $y = b$, the given DE has $y = 0$ and $y = b$ as solutions. More generally, if $y' = f(t, y)$ and $f(t, c) = 0$ for all t in some interval (I), the constant function $y = c$ on (I) is a solution of $y' = f(t, y)$ since $y' = 0$ for a constant function y.

To solve the logistic equation, we write it in the form

$$\frac{y'}{y(b - y)} = a.$$

Integrating both sides with respect to t we get

$$\int \frac{y'\mathrm{d}t}{y(b - y)} = at + C$$

which can, since $y'\mathrm{d}t = \mathrm{d}y$, be written as

$$\int \frac{\mathrm{d}y}{y(b - y)} = at + C.$$

Since, by partial fractions,

$$\frac{1}{y(b - y)} = \frac{1}{b}(\frac{1}{y} + \frac{1}{b - y})$$

we obtain

$$\frac{1}{b}(\ln|y| - \ln|b - y|) = at + C.$$

Multiplying both sides by b and exponentiating both sides to the base e, we get

$$\frac{|y|}{|b - y|} = \mathrm{e}^{bC}\mathrm{e}^{abt} = C_1\mathrm{e}^{abt},$$

where the arbitrary constant $C_1 = \pm e^{bC}$ can be determined by the initial condition (IC): $y(0) = y_0$ as

$$C_1 = \frac{|y_0|}{|b - y_0|}.$$

Two cases need to be discussed separately.

Case (I), $y_0 < b$: one has $C_1 = |\frac{y_0}{b-y_0}| = \frac{y_0}{b-y_0} > 0$. So that,

$$\frac{|y|}{|b-y|} = \left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt} > 0, \quad (t \in (I)).$$

From the above we derive $y/(b-y) = C_1 \mathrm{e}^{abt}$, and $y = (b-y)C_1 \mathrm{e}^{abt}$. This gives

$$y = \frac{bC_1 \mathrm{e}^{abt}}{1 + C_1 \mathrm{e}^{abt}} = \frac{b\left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt}}{1 + \left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt}}.$$

It shows that if $y_0 = 0$, one has the solution $y(t) = 0$. However, if $0 < y_0 < b$, one has the solution $0 < y(t) < b$, and as $t \to \infty$, $y(t) \to b$.

Case (II), $y_0 > b$: one has $C_1 = |\frac{y_0}{b-y_0}| = -\frac{y_0}{b-y_0} > 0$. So that,

$$\left|\frac{y}{b-y}\right| = \left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt} > 0, \quad (t \in (I)).$$

From the above we derive $y/(y-b) = \left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt}$, and $y = (y - b)\left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt}$. This gives

$$y = \frac{b\left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt}}{\left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt} - 1}.$$

It shows that if $y_0 > b$, one has the solution $y(t) > b$, and as $t \to \infty$, $y(t) \to b$.

It is derived that

- $y(t) = 0$ is an unstable equilibrium state of the system;
- $y(t) = b$ is a stable equilibrium state of the system.

4. Fundamental Existence and Uniqueness Theorem

If the function $f(x, y)$ together with its partial derivative with respect to y are continuous on the rectangle

$$R : |x - x_0| \leq a, \ |y - y_0| \leq b$$

there is a unique solution to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

defined on the interval $|x - x_0| < h$ where

$$h = \min(a, b/M), \quad M = \max|f(x,y)|, \ (x,y) \in R.$$

Note that this theorem indicates that a solution may not be defined for all x in the interval $|x - x_0| \le a$. For example, the function

$$y = \frac{bCe^{abx}}{1 + Ce^{abx}}$$

is solution to $y' = ay(b-y)$ but not defined when $1 + Ce^{abx} = 0$ even though $f(x,y) = ay(b-y$ satisfies the conditions of the theorem for all x, y.

The next example show why the condition on the partial derivative in the above theorem is necessary.

Consider the differential equation $y' = y^{1/3}$. Again $y = 0$ is a solution. Separating variables and integrating, we get

$$\int \frac{dy}{y^{1/3}} = x + C_1$$

which yields $y^{2/3} = 2x/3 + C$ and hence $y = \pm(2x/3 + C)^{3/2}$. Taking $C = 0$, we get the solution $y = (2x/3)^{3/2}, (x \ge 0)$ which along with the solution $y = 0$ satisfies $y(0) = 0$. So the initial value problem $y' = y^{1/3}$, $y(0) = 0$ does not have a unique solution. The reason this is so is due to the fact that $\frac{\partial f}{\partial y}(x,y) = 1/3y^{2/3}$ is not continuous when $y = 0$.

Many differential equations become linear or separable after a change of variable. We now give two examples of this.

5. Bernoulli Equation:

$$y' = p(x)y + q(x)y^n \quad (n \ne 1).$$

Note that $y = 0$ is a solution. To solve this equation, we set $u = y^\alpha$, where α is to be determined. Then, we have $u' = \alpha y^{\alpha-1}y'$, hence, our differential equation becomes

$$u'/\alpha = p(x)u + q(x)y^{\alpha+n-1}. \tag{2.5}$$

Now set $\alpha = 1 - n$. Thus, (2.5) is reduced to

$$u'/\alpha = p(x)u + q(x), \tag{2.6}$$

which is linear. We know how to solve this for u from which we get solve $u = y^{1-n}$ to get y.

6. Homogeneous Equation:

$$y' = F(y/x).$$

To solve this we let $u = y/x$ so that $y = xu$ and $y' = u+xu'$. Substituting for y, y' in our DE gives $u + xu' = F(u)$ which is a separable equation. Solving this for u gives y via $y = xu$. Note that $u = a$ is a solution of $xu' = F(u) - u$ whenever $F(a) = a$ and that this gives $y = ax$ as a solution of $y' = f(y/x)$.

Example. $y' = (x - y)/x + y$. This is a homogeneous equation since

$$\frac{x-y}{x+y} = \frac{1-y/x}{1+y/x}.$$

Setting $u = y/x$, our DE becomes

$$xu' + u = \frac{1-u}{1+u}$$

so that

$$xu' = \frac{1-u}{1+u} - u = \frac{1-2u-u^2}{1+u}.$$

Note that the right-hand side is zero if $u = -1\pm\sqrt{2}$. Separating variables and integrating with respect to x, we get

$$\int \frac{(1+u)du}{1-2u-u^2} = \ln|x| + C_1$$

which in turn gives

$$(-1/2)\ln|1-2u-u^2| = \ln|x| + C_1.$$

Exponentiating, we get

$$\frac{1}{\sqrt{|1-2u-u^2|}} = e^{C_1}|x|.$$

Squaring both sides and taking reciprocals, we get

$$u^2 + 2u - 1 = C/x^2$$

with $C = \pm 1/e^{2C_1}$. This equation can be solved for u using the quadratic formula. If x_0, y_0 are given with $x_0 \neq 0$ and $u_0 = y_0/x_0 \neq -1$ there is, by the fundamental, existence and uniqueness theorem,a unique solution with $u(x_0) = y_0$. For example, if $x_0 = 1, y_0 = 2$, we have $C = 7$ and hence

$$u^2 + 2u - 1 = 7/x^2$$

Solving for u, we get

$$u = -1 + \sqrt{2 + 7/x^2}$$

where the positive sign in the quadratic formula was chosen to make $u = 2, x = 1$ a solution. Hence

$$y = -x + x\sqrt{2 + 7/x^2} = -x + \sqrt{2x^2 + 7}$$

is the solution to the initial value problem

$$y' = \frac{x - y}{x + y}, \quad y(1) = 2$$

for $x > 0$ and one can easily check that it is a solution for all x. Moreover, using the fundamental uniqueness, it can be shown that it is the only solution defined for all x.

PART (III): EXACT EQUATION AND INTEGRATING FACTOR — NONLINEAR EQUATIONS (2)

7. Exact Equations.

By a region of the xy-plane we mean a connected open subset of the plane. The differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is said to be exact on a region (R) if there is a function $F(x,y)$ defined on (R) such that

$$\frac{\partial F}{\partial x} = M(x,y); \quad \frac{\partial F}{\partial y} = N(x,y)$$

In this case, if M, N are continuously differentiable on (R) we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2.7}$$

Conversely, it can be shown that condition (2.7) is also sufficient for the exactness of the given DE on (R) providing that (R) is simply connected, .i.e., has no "holes".

The exact equations are solvable. In fact, suppose $y(x)$ is its solution. Then one can write:

$$M\left[x, y(x)\right] + N\left[x, y(x)\right]\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = \frac{d}{dx}F\left[x, y(x)\right] = 0.$$

It follows that

$$F\left[x, y(x)\right] = C,$$

where C is an arbitrary constant. This is an implicit form of the solution $y(x)$. Hence, the function $F(x,y)$, if it is found, will give a family of the solutions of the given DE. The curves $F(x,y)=C$ are called integral curves of the given DE.

Example 1. $2x^2y\frac{dy}{dx}+2xy^2+1=0$. Here $M=2xy^2+1$, $N=2x^2y$ and $R=\mathcal{R}^2$, the whole xy-plane. The equation is exact on $\mathcal{R}^2$ since $\mathcal{R}^2$ is simply connected and

$$\frac{\partial M}{\partial y}=4xy=\frac{\partial N}{\partial x}.$$

To find F we have to solve the partial differential equations

$$\frac{\partial F}{\partial x}=2xy^2+1, \quad \frac{\partial F}{\partial y}=2x^2y.$$

If we integrate the first equation with respect to x holding y fixed, we get

$$F(x,y)=x^2y^2+x+\phi(y).$$

Differentiating this equation with respect to y gives

$$\frac{\partial F}{\partial y}=2x^2y+\phi'(y)=2x^2y$$

using the second equation. Hence $\phi'(y)=0$ and $\phi(y)$ is a constant function. The solutions of our DE in implicit form is $x^2y^2+x=C$.

Example 2. We have already solved the homogeneous DE

$$\frac{dy}{dx}=\frac{x-y}{x+y}.$$

This equation can be written in the form

$$y-x+(x+y)\frac{dy}{dx}=0$$

which is an exact equation. In this case, the solution in implicit form is $x(y-x)+y(x+y)=C$, i.e., $y^2+2xy-x^2=C$.

8. Theorem.

If $F(x,y)$ is homogeneous of degree n then

$$x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}=nF(x,y).$$

Proof. The function F is homogeneous of degree n if $F(tx, ty) = t^n F(x, y)$. Differentiating this with respect to t and setting $t = 1$ yields the result. **QED**

9. Integrating Factors.

If the differential equation $M + Ny' = 0$ is not exact it can sometimes be made exact by multiplying it by a continuously differentiable function $\mu(x, y)$. Such a function is called an *integrating factor*. An integrating factor μ satisfies the PDE $\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x}$ which can be written in the form

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}.$$

This equation can be simplified in special cases, two of which we treat next.

- μ is a function of x only. This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = p(x)$$

  is a function of x only in which case $\mu' = p(x)\mu$.

- μ is a function of y only. This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = q(y)$$

  is a function of y only in which case $\mu' = -q(y)\mu$.

- $\mu = P(x)Q(y)$. This happens if and only if

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = p(x)N - q(y)M, \tag{2.8}$$

  where

$$p(x) = \frac{P'(x)}{P(x)}, \quad q(y) = \frac{Q'(y)}{Q(y)}.$$

  If the system really permits the functions $p(x), q(y)$, such that (2.8) hold, then we can derive

$$P(x) = \pm e^{\int p(x)\mathrm{d}x}; \qquad Q(y) = \pm e^{\int q(y)\mathrm{d}y}.$$

Example 1. $2x^2 + y + (x^2y - x)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2}{x}$$

so that there is an integrating factor μ which is a function of x only which satisfies $\mu' = -2\mu/x$. Hence $\mu = 1/x^2$ is an integrating factor and $2 + y/x^2 + (y - 1/x)y' = 0$ is an exact equation whose general solution is $2x - y/x + y^2/2 = C$ or $2x^2 - y + xy^2/2 = Cx$.

Example 2. $y + (2x - ye^y)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y}$$

so that there is an integrating factor which is a function of y only which satisfies $\mu' = 1/y$. Hence y is an integrating factor and $y^2 + (2xy - y^2e^y)y' = 0$ is an exact equation with general solution $xy^2 + (-y^2 + 2y - 2)e^y = C$.

A word of caution is in order here. The solutions of the exact DE obtained by multiplying by the integrating factor may have solutions which are not solutions of the original DE. This is due to the fact that μ may be zero and one will have to possibly exclude those solutions where μ vanishes. However, this is not the case for the above Example 2.

PART (IV): CHANGE OF VARIABLES — NONLINEAR EQUATIONS (3)

10. Change of Variables.

Sometimes it is possible by means of a change of variable to transform a DE into one of the known types. For example, homogeneous equations can be transformed into separable equations and Bernoulli equations can be transformed into linear equations. The same idea can be applied to some other types of equations, as described as follows.

10.1 $y' = f(ax + by),\ b \neq 0$

Here, if we make the substitution $u = ax+by$ the differential equation becomes

$$\frac{du}{dx} = bf(u) + a$$

which is separable.

Example 1. The DE $y' = 1 + \sqrt{y - x}$ becomes $u' = \sqrt{u}$ after the change of variable $u = y - x$.

10.2 $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

Here, we assume that $a_1x+b_1y+c_1 = 0$, $a_2x+b_2y+c_2 = 0$ are distinct lines meeting in the point (x_0, y_0). The above DE can be written in the form

$$\frac{dy}{dx} = \frac{a_1(x - x_0) + b_1(y - y_0)}{a_2(x - x_0) + b_2(y - y_0)}$$

which yields the DE

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

after the change of variables $X = x - x_0$, $Y = y - y_0$.

10.3 Riccati equation: $y' = p(x)y + q(x)y^2 + r(x)$

Suppose that $u = u(x)$ is a solution of this DE and make the change of variables $y = u + 1/v$. Then $y' = u' - v'/v^2$ and the DE becomes

$$\begin{aligned} u' - v'/v^2 &= p(x)(u + 1/v) + q(x)(u^2 + 2u/v + 1/v^2) + r(x) \\ &= p(x)u + q(x)u^2 + r(x) + (p(x) + 2uq(x))/v + q(x)/v^2 \end{aligned}$$

from which we get $v' + (p(x) + 2uq(x))v = -q(x)$, a linear equation.

Example 2. $y' = 1 + x^2 - y^2$ has the solution $y = x$ and the change of variable $y = x + 1/v$ transforms the equation into $v' + 2xv = 1$.

PART (V): SOME APPLICATIONS

We now give a few applications of differential equations.

11. Orthogonal Trajectories.

An important application of first order DE's is to the computation of the orthogonal trajectories of a family of curves $f(x,y,C)=0$. An orthogonal trajectory of this family is a curve that, at each point of intersection with a member of the given family, intersects that member orthogonally. To find the orthogonal trajectories, we may derive the ODE, whose solutions are described by these trajectories. For this purpose, we are going first to derive the ODE, whose solutions have the implicit form, $f(x,y,C)=0$. In doing so, we differentiate $f(x,y,C)=0$ implicitly with respect to x we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}y'=0$$

from which we get

$$y'=-\frac{f_x(x,y,C)}{f_y(x,y,C)}.$$

Now we solve for $C=C(x,y)$ from the equation $f(x,y,C)=0$, which specifies the curve passing through the point (x,y). We substitute $C(x,y)$ in the above formula for y'. This gives the equation:

$$y'=g(x,y)=-\frac{f_x\Big[x,y,C(x,y)\Big]}{f_y\Big[x,y,C(x,y)\Big]}.$$

Note that $y'(x)$ yields the slope of the tangent line at the point (x,y) of a curve of the given family passing through (x,y). The slope of the orthogonal trajectory at the passing point (x, y) must be

$$y'(x) = -\frac{1}{g(x,y)}.$$

Therefore, the ODE governing the orthogonal trajectories is derived as

$$y' = \frac{f_y\Big[x, y, C(x,y)\Big]}{f_x\Big[x, y, C(x,y)\Big]}.$$

Example 3. Let us find the orthogonal trajectories of the family $x^2 + y^2 = Cx$, the family of circles with center on the x-axis and passing through the origin. Here

$$2x + 2yy' = C = \frac{x^2 + y^2}{x}$$

from which, we derive the ODE: $y' = g(x, y) = (y^2 - x^2)/2xy$. Then the ODE governing the orthogonal trajectories can be written as

$$y' = -\frac{1}{g(x,y)},$$

or,

$$y' = 2xy/(x^2 - y^2).$$

The above can be re-written in the form:

$$2xy + (y^2 - x^2)y' = 0.$$

If we let $M = 2xy$, $N = y^2 - x^2$ we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{4x}{2xy} = \frac{2}{y}$$

so that we have an integrating factor μ which is a function of y. We have $\mu' = -2\mu/y$ from which $\mu = 1/y^2$. Multiplying the DE for the orthogonal trajectories by $1/y^2$ we get

$$\frac{2x}{y} + \left(1 - \frac{x^2}{y^2}\right) y' = 0.$$

Solving $\frac{\partial F}{\partial x} = 2x/y$, $\frac{\partial F}{\partial y} = 1 - x^2/y^2$ for F yields $F(x, y) = x^2/y + y$ from which the orthogonal trajectories are $x^2/y + y = C$, i.e., $x^2 + y^2 = Cy$. This is the family of circles with center on the y-axis and passing through the origin. Note that the line $y = 0$ is also an orthogonal trajectory that was not found by the above procedure. This is due to the fact that the integrating factor was $1/y^2$ which is not defined if $y = 0$ so we had to work in a region which does not cut the x-axis, e.g., $y > 0$ or $y < 0$.

12. Falling Bodies with Air Resistance

Let x be the height at time t of a body of mass m falling under the influence of gravity. If g is the force of gravity and $b\,v$ is the force on the body due to air resistance, Newton's Second Law of Motion gives the DE

$$m\frac{dv}{dt} = mg - bv$$

where $v = \frac{dx}{dt}$. This DE has the general solution

$$v(t) = \frac{mg}{b} + Be^{-bt/m}.$$

The limit of $v(t)$ as $t \to \infty$ is mg/b, the terminal velocity of the falling body. Integrating once more, we get

$$x(t) = A + \frac{mg\,t}{b} - \frac{mB}{b}e^{-bt/m}.$$

13. Mixing Problems

Suppose that a tank is being filled with brine at the rate of a units of volume per second and at the same time b units of volume per second are pumped out. If the concentration of the brine coming in is c units of weight per unit of volume. If at time $t = t_0$ the volume of brine in the tank is V_0 and contains x_0 units of weight of salt, what is the quantity of salt in the tank at any time t, assuming that the tank is well mixed?

If x is the quantity of salt at any time t, we have ac units of weight of salt coming in per second and

$$b\frac{x(t)}{V(t)} = \frac{bx}{V_0 + (a-b)(t-t_0)}$$

units of weight of salt going out. Hence

$$\frac{dx}{dt} = ac - \frac{bx}{V_0 + (a-b)(t-t_0)},$$

a linear equation. If $a = b$ it has the solution

$$x(t) = cV_0 + (x_0 - cV_0)e^{-a(t-t_0)/V_0}.$$

As a numerical example, suppose $a = b = 1$ liter/min, $c = 1$ grams/liter, $V_0 = 1000$ liters, $x_0 = 0$ and $t_0 = 0$. Then

$$x(t) = 1000(1 - e^{-.001t})$$

is the quantity of salt in the tank at any time t. Suppose that after 100 minutes the tank springs a leak letting out an additional liter of brine per minute. To find out how much salt is in the tank 12 hours after the leak begins we use the DE

$$\frac{dx}{dt} = 1 - \frac{2x}{1000-(t-100)} = 1 - \frac{2}{1100-t}x.$$

This equation has the general solution

$$x(t) = (1100-t)^{-1} + C(1100-t)^2.$$

Using $x(100) = 1000(1-e^{-.1}) = 95.16$, we find $C = -9.048 \times 10^{-4}$ and $x(820) = 177.1$. When $t = 1100$ the tank is empty and the differential equation is no a valid description of the physical process. The concentration at time $100 < t < 1100$ is

$$\frac{x(t)}{1100-t} = 1 + C(1100-t)$$

which converges to 1 as t tends to 1100.

14. Heating and Cooling Problems

Newton's Law of Cooling states that the rate of change of the temperature of a cooling body is proportional to the difference between its temperature T and the temperature of its surrounding medium. Assuming the surroundings maintain a constant temperature T_s, we obtain the differential equation

$$\frac{dT}{dt} = -k(T-T_s),$$

where k is a constant. This is a linear DE with solution

$$T = T_s + Ce^{-kt}.$$

If $T(0) = T_0$ then $C = T_0 - T_s$ and

$$T = T_s + (T_0 - T_s)e^{-kt}.$$

As an example consider the problem of determining the time of death of a healthy person who died in his home some time before noon when his body was 70 degrees. If his body cooled another 5 degrees in 2 hours when did he die, assuming that the room was a constant 60 degrees. Taking noon as $t = 0$ we have $T_0 = 70$. Since $T_s = 60$, we get $65 - 60 = 10e^{-2k}$ from which $k = \ln(2)/2$. To determine the time of death we use the equation $98.6 - 60 = 10e^{-kt}$ which gives $t = -\ln(3.86)/k = -2\ln(3.86)/\ln(2) = -3.90$. Hence the time of death was $8:06$ AM.

15. Radioactive Decay

A radioactive substance decays at a rate proportional to the amount of substance present. If x is the amount at time t we have

$$\frac{dx}{dt} = -kx,$$

where k is a constant. The solution of the DE is $x = x(0)e^{-kt}$. If c is the half-life of the substance we have by definition

$$x(0)/2 = x(0)e^{-kc}$$

which gives $k = \ln(2)/c$.

PART (VI)*: GEOMETRICAL APPROACHES — NONLINEAR EQUATIONS (4)

16. Definitions and Basic Concepts

16.1 Directional Field

A plot of short line segments drawn at various points in the (x, y) plane showing the slope of the solution curve there is called *direction field* for the DE.

16.2 Integral Curves

The family of curves in the (x, y) plane, that represent all solutions of DE is called the *integral curves*.

16.3 Autonomous Systems

The first order DE

$$dy/dx = f(y)$$

is called autonomous, since the independent variable does not appear explicitly. The isoclines are made up of horizonal lines $y = m$, along which the slope of directional fields is the constant, $y' = f(m)$.

16.4 Equilibrium Points

The DE has the constant solution $y = y_0$, if and only if $f(y_0) = 0$. These values of y_0 are the **equilibrium points** or **stationary points** of the DE. $y = y_0$ is called a **source** if $f(y)$ changes sign from - to + as y increases from just below $y = y_0$ to just above $y = y_0$ and is called a **sink** if $f(y)$ changes sign from + to - as y increases from just below $y = y_0$ to just above $y = y_0$; it is called a **node** if there is no change in sign. Solutions $y(t)$ of the DE appear to be attracted by the line $y = y_0$, if y_0 is a sink and move away from the line $y = y_0$, if y_0 is a source.

17. Phase Line Analysis

The y-axis on which is plotted the equilibrium points of the DE with arrows between these points to indicate when the solution y is increasing or decreasing is called the phase line of the DE. The autonomous DE

$$dy/dx = 2y - y^2$$

has 0 and 1 as equilibrium points. The point $y = 0$ is a source and $y = 2$ is a sink (see Fig.2.1). This DE is a logistic model for a population having 2 as the size of a stable population. The equation

$$dy/dx = -y(2 - y)(3 - y)$$

has three equilibrium states: $y = 0, 2, 3$. Among them, $y = 0, 3$ are the sink, while $y = 2$ is the source (see Fig.2.2). The equation

$$dy/dx = -y(2 - y)^2$$

has two equilibrium states: $y = 0, 2$. The point $y = 0$ is a sink, while $y = 2$ is a node (see Fig.2.3). The sink is stable, source is unstable, whereas the node is semi-stable. The node point of the equation $y = f(y)$ can either disappear, or split into one sink and one source, when the equation is perturbed with a small amount ε and becomes: $y = f(y) + \varepsilon$.

18. Bifurcation Diagram

Some dynamical system contains a parameter Λ, such as

$$y' = f(y, \Lambda).$$

Then the characteristics of its equilibrium states, such as their number and nature, depends on the value of Λ. Some times, through a special value of $\Lambda = \Lambda_*$, these characteristics of equilibrium states may change. This $\Lambda = \Lambda_*$ is called the **bifurcation point**.

Example 1. For the logistic population growth model, if the population is reduced at a constant rate $s > 0$, the DE becomes

$$dy/dx = 2y - y^2 - s$$

which has a source at the larger of the two roots of the equation

$$y^2 - 2y + s = 0$$

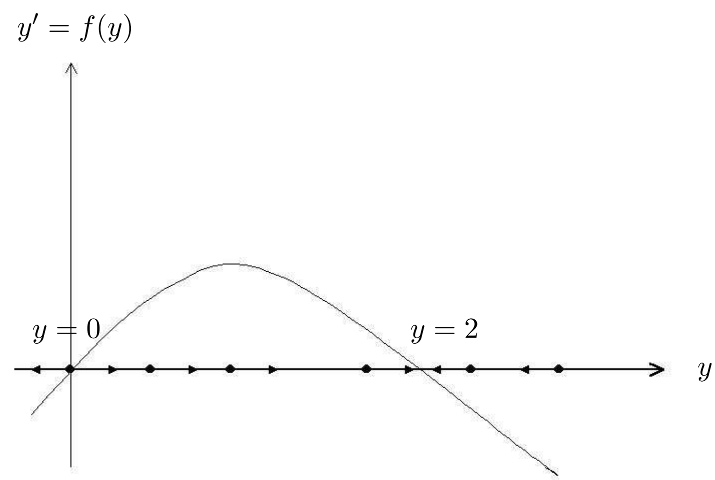

Figure 2.1. Sketch of the phase line for the equation $dy/dx = 2y - y^2$, in which $y = 0$ is a source, $y = 2$ is a sink.

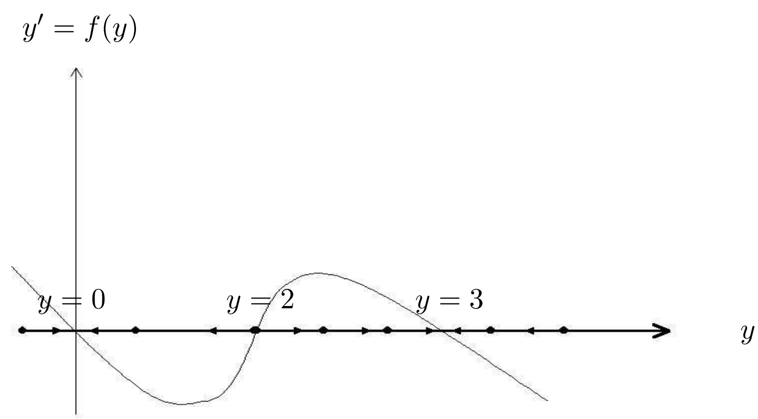

Figure 2.2. Sketch of the phase line for the equation $dy/dx = -y(2-y)(3-y)$, in which $y = 0, 3$ is a sink, $y = 2$ is a source.

for $s < 2$. If $s > 2$ there is no equilibrium point and the population dies out as y is always decreasing. The point s=2 is called a bifurcation point of the DE.

Example 2. Chemical Reaction Model. One has the DE

$$dy/dx = -ay\left[y^2 - \frac{R - R_c}{a}\right],$$

where

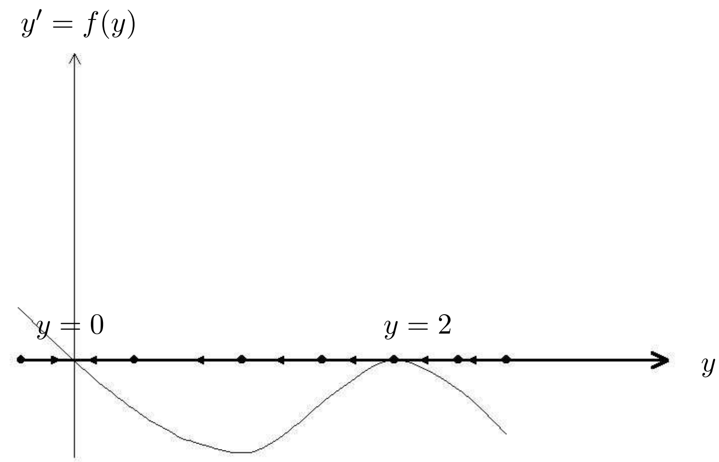

Figure 2.3. Sketch of the phase line for the equation $dy/dx = -y(2-y)^2$, in which the point $y = 0$ is a sink, while $y = 2$ is a node.

- y is the concentration of species A;
- R is the concentration of some chemical element,

and (a, R_c) are constants (fixed). It is derived that

- If $R < R_c$, the system has one equilibrium state $y = 0$, which is stable;
- If $R > R_c$, the system has three equilibrium states: $y = 0$, which is now unstable, and $y = \pm\sqrt{\frac{R-R_c}{a}}$, which are stable.

For this system, $R = R_c$ is the bifurcation point.

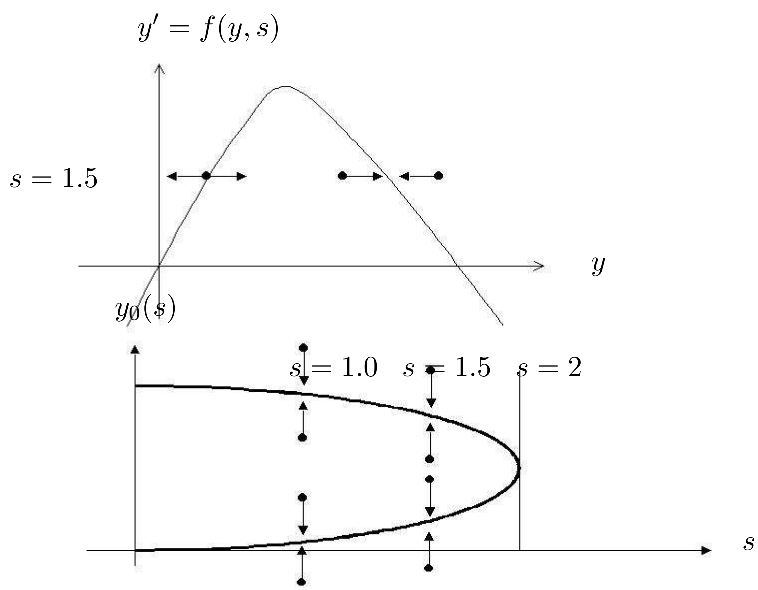

Figure 2.4. Sketch of the bifurcation diagram of the equation $dy/dx = y(2 - y) - s$, in which the point $s = 2$ is the bifurcation point.

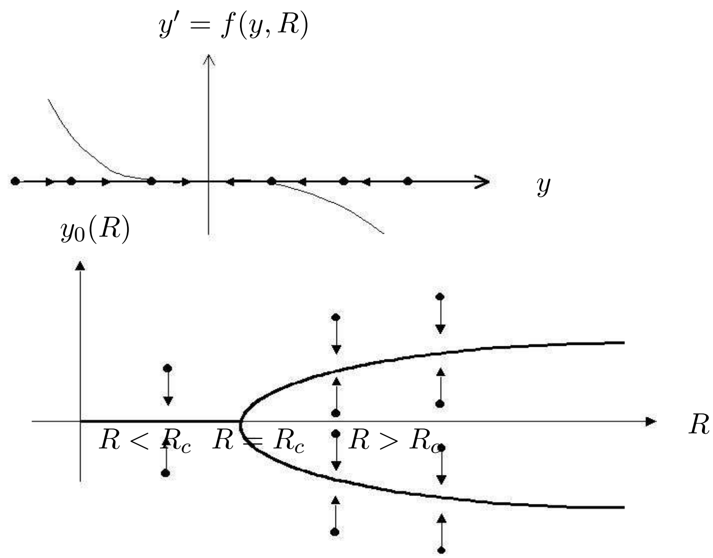

Figure 2.5. Sketch of the bifurcation diagram of the equation $dy/dx = -ay\left[y^2 - \frac{R-R_c}{a}\right]$, in which the point $R = R_c$ is the bifurcation point.

PART (VII): NUMERICAL APPROACH AND APPROXIMATIONS

19. Euler's Method

In this section we discuss methods for obtaining a numerical solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

at equally spaced points $x_0, x_1, x_2, \ldots, x_N = p, \ldots$ where $x_{n+1} - x_n = h > 0$ is called the step size. In general, the smaller the value of the better the approximations will be but the number of steps required will be larger. We begin by integrating $y' = f(x, y)$ between x_n and x_{n+1}. If $y(x) = \phi(x)$, this gives

$$\phi(x_{n+1}) = \phi(x_n) + \int_{x_n}^{x_{n+1}} f(t, \phi(t))dt.$$

As a first estimate of the integrand we use the value of $f(t, \phi(t))$ at the lower limit x_n, namely $f(x_n, \phi(x_n))$. Now, assuming that we have already found an estimate y_n for $\phi(x_n)$, we get the estimate

$$y_{n+1} = y_n + hf(x_n, y_n)$$

for $\phi(x_{n+1})$. It can be shown that

$$|y_n - \phi(x_n)| \leq Ch,$$

where C is a constant which depends on p.

20. Improved Euler's Method

The Euler method can be improved if we use the trapezoidal rule for estimating the above integral. Namely,

$$\int_a^b F(x)dx = \frac{1}{2}(F(a) + F(b))(b-a).$$

This leads to the estimate

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_{n+1}, y_{n+1})).$$

If we now use the Euler approximation y_{n+1} to compute $f(x_{n+1}, y_{n+1})$, we get

$$y_{n+1} = y_n + \frac{h}{2}(f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n)).$$

This is known as the improved Euler method. It can be shown that

$$|y_n - \phi(x_n)| \leq Ch^2.$$

In general, if y_n is an approximation for $\phi(x_n)$ such that

$$|y_n - \phi(x_n)| \leq Ch^p,$$

we say that the approximation is of order p. Thus the Euler method is first order and the improved Euler is second order.

21. Higher Order Methods

On can obtain higher order approximations by using better approximations for $F(t) = f(t, \phi(t))$ on the interval $[x_n, x_{n+1}]$. For example, the Taylor series approximation

$$F(t) = F(x_n)+F'(x_n)(t-x_n)+F''(x_n)(t-x_n)^2/2+\cdots+F^{(p-1)}(x_n)(t-x_n)^{p-1}/(p-1)!$$

yields the approximation

$$y_{n+1} = y_n + hf_1(x_x, y_n) + \frac{h^2}{2}f_2(x_n, y_n) + \cdots + \frac{h^p}{p!}f_{p-1}(x_n, y_n),$$

where

$$f_k(x_n, y_n) = F^{(k-1)}(x_n) = \left[\frac{\partial}{\partial x} + f(x,y)\frac{\partial}{\partial y}\right]^{(k-1)} f(x_n, y_n).$$

It can be show that this approximation is of order p. However it is computationally intensive as one has to compute higher derivatives.

In the case $p = 2$ this formula was simplified by Runge and Kutta to give the second order midpoint approximation

$$y_{n+1} = y_n + hf\left[x_n + \frac{h}{2},\ y_n + \frac{h}{2}f(x_n, y_n)\right].$$

In the case $p = 4$ they obtained the 4-th order approximation

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2}), \\ k_3 &= hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_2}{2}), \\ k_4 &= hf(x_n + h, y_n + k_3). \end{aligned} \tag{2.9}$$

Computationally, it is much simpler than the 4-th order Taylor series approximation from which it is derived.

4$^{(*)}$ Picard Iteration

We assume that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on the rectangle

$$R: \ |x - x_0| \leq a, \ |y - y_0| \leq b$$

Then $|f(x, y)| \leq M$, $|\frac{\partial f}{\partial y}(x, y)| \leq L$ on R. The initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ is equivalent to the integral equation

$$y = y_0 + \int_{x_0}^{x} f(t, y(t))dt.$$

Let the righthand side of the above equation be denoted by $T(y)$. Then our problem is to find a solution to $y = T(y)$ which is a fixed point problem. To solve this problem we take as an initial approximation to y the constant function $y_0(x) = y_0$ and consider the iterations $y_n = T^n(y_0)$. The function y_n is called the n-th Picard iteration of y_0. For example, for the initial value problem $y' = x + y^2$, $y(0) = 1$ we have

$$y_1(x) = 1 + \int_0^x (t + 1)dt = 1 + x + x^2/2$$

$$y_2(x) = 1 + \int_0^x (t + (1 + t + t^2/2)^2)dt = 1 + x + 3x^2/2 + 2x^3/3 + x^4/4 + x^5/20.$$

Contrary to the power series approximations we can determine just how good the Picard iterations approximate y. In fact, we will see that the Picard iterations converge to a solution of our initial value problem. More precisely we have the following result:

4.1 Theorem of Existence and Uniqueness of Solution for IVP

The Picard iterations $y_n = T^n(y_0)$ converge to a solution y of $y' = f(x,y)$, $y(x_0) = y_0$ on the interval $|x-x_0| \leq h = \min(a, b/M)$. Moreover

$$|y(x) - y_n(x)| \leq (M/L)e^{hL}(Lh)^{n+1}/(n+1)!$$

for $|x - x_0| \leq h$ and the solution y is unique on this interval.

Proof. We have

$$|y_1 - y_0| = |\int_{x_0}^{x} f(t, y_0)| \leq M|x - x_0|$$

since $|f(x,y)| \leq M$ on R. Now $|y_1 - y_0| \leq b$ if $|x - x_0| \leq h$. So $(x, y_1(x))$ is in R if $|x-x_0| \leq h$. Similarly, one can show inductively that $(x, y_n(x))$ is in R if $|x - x_0| \leq h$. Using the fact that, by the mean value theorem for derivatives,

$$|f(x,z) - f(x,w| \leq L|z - w|$$

for all $(x,w), (x,z)$ in R, we obtain

$$|y_2 - y_1| = |\int_{x_0}^{x} (f(t,y_1) - f(t,y_0)| \leq ML|x - x_0|^2/2,$$

$$|y_3 - y_2| = |\int_{x_0}^{x} (f(t,y_2) - f(t,y_1)| \leq ML^2|x - x_0|^3/6$$

and by induction $|y_n - y_{n-1}| \leq ML^{n-1}|x - x_0|^n/n!$. Since the series $\sum_1^\infty |y_n - y_{n-1}|$ is bounded above term by term by the convergent series $(M/L)\sum_1^\infty (L|x-x_0|)^n/n!$, its n-th partial sum $y_n - y_0$ converges, which gives the convergence of y_n to a function y. Now since

$$y = y_0 + (y_1 - y_0) + \cdots + (y_n - y_{n-1}) + \sum_{i=n+1}^{\infty} (y_i - y_{i-1})$$

we obtain

$$|y - y_n| \leq \sum_{i=n+1}^{\infty} (M/L)(L(|x - x_0|)^i/i! \leq (M/L)\frac{(Lh)^{n+1}}{(n+1)!}e^{hL}.$$

For the uniqueness, suppose $T(z) = z$ with $(x, z(x)$ in R for $|x-x_0| \leq h$. Then

$$y(x) - z(x) = \int_{x_0}^{x} (f(t, y(x)) - f(t, z(x))dt.$$

If $|y(x) - z(x)| \leq A$ for $x - x_0| \leq h$ we then obtain as above

$$|y(x) - z(x)| \leq AL|x - x_0|.$$

Now using this estimate, repeat the above to get

$$|y(x) - z(x)| \leq AL^2|x - x_0|^2/2.$$

Using induction we get that

$$|y(x) - z(x)| \leq AL^n|x - x_0|^n/n!$$

which converges to zero for all x. Hence $y = z$. **QED**

The key ingredient in the proof is the **Lipschitz Condition**

$$|f(x, y) - f(x, z)| \leq L|y - z|.$$

If $f(x, y)$ is continuous for $|x - x_0| \leq a$ and all y and satisfies the above Lipschitz condition in this strip the above proof gives the existence and uniqueness of the solution to the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ on the interval $|x - x_0| \leq a$.

Chapter 3

N-TH ORDER DIFFERENTIAL EQUATIONS

PART (I): THE FUNDAMENTAL EXISTENCE AND UNIQUENESS THEOREM

In this lecture we will state and sketch the proof of the fundamental existence and uniqueness theorem for the n-th order DE

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

The starting point is to convert this DE into a system of first order DE'. Let $y_1 = y, y_2 = y', \dots y^{(n-1)} = y_n$. Then the above DE is equivalent to the system

$$\begin{aligned} \tfrac{dy_1}{dx} &= y_2 \\ \tfrac{dy_2}{dx} &= y_3 \\ &\vdots \\ \tfrac{dy_n}{dx} &= f(x, y_1, y_2, \dots, y_n). \end{aligned} \tag{3.1}$$

More generally let us consider the system

$$\begin{aligned} \tfrac{dy_1}{dx} &= f_1(x, y_1, y_2, \dots, y_n) \\ \tfrac{dy_2}{dx} &= f_2(x, y_1, y_2, \dots, y_n) \\ &\vdots \\ \tfrac{dy_n}{dx} &= f_n(x, y_1, y_2, \dots, y_n). \end{aligned} \tag{3.2}$$

If we let $Y = (y_1, y_2, \dots, y_n), F(x, Y) = \Big\{ f_1(x, Y), f_2(x, Y), \dots, f_n(x, Y) \Big\}$ and $\frac{dY}{dx} = (\frac{dy_1}{dx}, \frac{dy_2}{dx}, \dots, \frac{dy_n}{dx})$, the system becomes

$$\frac{dY}{dx} = F(x, Y).$$

1. Theorem of Existence and Uniqueness (I)

If $f_i(x, y_1, \ldots, y_n)$ and $\frac{\partial f_i}{\partial y_j}$ are continuous on the $n+1$-dimensional box

$$R : |x - x_0| < a,\ |y_i - c_i| < b, (1 \le i \le n)$$

for $1 \le i, j \le n$ with $|f_i(x, y)| \le M$ and

$$\left|\frac{\partial f_i}{\partial y_1}\right| + \left|\frac{\partial f_i}{\partial y_2}\right| + \ldots \left|\frac{\partial f_i}{\partial y_n}\right| < L$$

on R for all i, the initial value problem

$$\frac{dY}{dx} = F(x, Y), \quad Y(x_0) = (c_1, c_2, \ldots, c_n)$$

has a unique solution on the interval $|x - x_0| \le h = \min(a, b/M)$.

The proof is exactly the same as for the proof for $n = 1$ if we use the following Lemma in place of the mean value theorem.

1.1 Lemma

If $f(x_1, x_2, \ldots x_n)$ and its partial derivatives are continuous on an n-dimensional box R, then for any $a, b \in R$ we have

$$|f(a) - f(b)| \le \left(\left|\frac{\partial f}{\partial x_1}(c)\right| + \cdots + \left|\frac{\partial f}{\partial x_n}(c)\right|\right) |a - b|$$

where c is a point on the line between a and b and $|(x_1, \ldots, x_n)| = \max(|x_1|, \ldots, |x_n|)$.

The lemma is proved by applying the mean value theorem to the function $G(t) = f(ta + (1 - t)b)$. This gives

$$G(1) - G(0) = G'(c)$$

for some c between 0 and 1. The lemma follows from the fact that

$$G'(x) = \frac{\partial f}{\partial x_1}(a_1 - b_1) + \cdots + \frac{\partial f}{\partial x_n}(a_n - b_n).$$

The Picard iterations $Y_k(x)$ defined by

$$Y_0(x) = Y_0 = (c_1, \ldots, c_n), \quad Y_{k+1}(x) = Y_0 + \int_{x_0}^{x} F(t, Y_k(t))dt,$$

converge to the unique solution Y and

$$|Y(x) - Y_k(x)| \le (M/L)e^{hL}h^{k+1}/(k + 1)!.$$

If $f_1(x, y_1, \ldots, y_)$, $\frac{\partial f_i}{\partial y_j}$ are continuous in the strip $|x - x_0| \leq a$ and there is an L such that

$$|f(x, Y) - f(x, Z)| \leq L|Y - Z|$$

then h can be taken to be a and $M = max|f(x, Y_0)|$. This happens in the important special case

$$f_i(x, y_1, \ldots, y_n) = a_{i1}(x)y_1 + \cdots + a_{in}(x)y_n + b_i(x).$$

As a corollary of the above theorem we get the following fundamental theorem for n-th order DE's.

2. Theorem of Existence and Uniqueness (II)

If $f(x, y_1, \ldots, y_n)$ and $\frac{\partial f}{\partial y_j}$ are continuous on the box

$$R : |x - x_0| \leq a, \ |y_i - c_i| \leq b \ (1 \leq i \leq n)$$

and $|f(x, y_1, \ldots, y_n)| \leq M$ on R, then the initial value problem

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad y^{i-1}(x_0) = c_i \ (1 \leq 1 \leq n)$$

has a unique solution on the interval $|x - x_0| \leq h = \max(a, b/M)$.

Another important application is to the n-th order linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x).$$

In this case $f_1 = y_2, \ f_2 = y_3, \ f_n = p_1(x)y_1 + \cdots p_n(x)y_n + q(x)$ where $p_i(x) = a_{n-i}(x)/a_0(x)$, $q(x) = -b(x)/a_0(x)$.

3. Theorem of Existence and Uniqueness (III)

If $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous on an interval I and $a_0(x) \neq 0$ on I then, for any $x_0 \in I$, that is not an endpoint of I, and any scalars $c_1, c_2, \ldots, c_n$, the initial value problem

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x), \quad y^{i-1}(x_0) = c_i \ (1 \leq 1 \leq n)$$

has a unique solution on the interval I.

PART (II): BASIC THEORY OF LINEAR EQUATIONS

In this lecture we give an introduction to several methods for solving higher order differential equations. Most of what we say will apply to the linear case as there are relatively few non-numerical methods for solving nonlinear equations. There are two important cases however where the DE can be reduced to one of lower degree.

3.1 Case (I)

DE has the form:

$$y^{(n)} = f(x, y', y'', \ldots, y^{(n-1)})$$

where on the right-hand side the variable y does not appear. In this case, setting $z = y'$ leads to the DE

$$z^{(n-1)} = f(x, z, z', \ldots, z^{(n-2)})$$

which is of degree $n-1$. If this can be solved then one obtains y by integration with respect to x.

For example, consider the DE $y'' = (y')^2$. Then, setting $z = y'$, we get the DE $z' = z^2$ which is a separable first order equation for z. Solving it we get $z = -1/(x+C)$ or $z = 0$ from which $y = -\log(x+C) + D$ or $y = C$. The reader will easily verify that there is exactly one of these solutions which satisfies the initial condition $y(x_0) = y_0$, $y'(x_0) = y'_0$ for any choice of x_0, y_0, y'_0 which confirms that it is the general solution since the fundamental theorem guarantees a unique solution.

3.2 Case (II)

DE has the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

where the independent variable x does not appear explicitly on the right-hand side of the equation. Here we again set $z = y'$ but try for a solution z as a function of y. Then, using the fact that $\frac{d}{dx} = z\frac{d}{dy}$, we get the DE

$$\left(z\frac{d}{dy}\right)^{n-1}(z) = f\left(y, z, z\frac{dz}{dy}, \ldots, (z\frac{d}{dy})^n(z)\right)$$

which is of degree $n-1$. For example, the DE $y'' = (y')^2$ is of this type and we get the DE

$$z\frac{dz}{dy} = z^2$$

which has the solution $z = Ce^y$. Hence $y' = Ce^y$ from which $-e^{-y} = Cx + D$. This gives $y = -\log(-Cx - D)$ as the general solution which is in agreement with what we did previously.

4. Linear Equations

4.1 Basic Concepts and General Properties

Let us now go to linear equations. The general form is

$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x).$$

The function L is called a *differential operator*. The characteristic feature of L is that

$$L(a_1y_1 + a_2y_2) = a_1L(y_1) + a_2L(y_2).$$

Such a function L is what we call a *linear operator*. Moreover, if

$$L_1(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$$

$$L_2(y) = b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_n(x)y$$

and $p_1(x), p_2(x)$ are functions of x the function $p_1L_1 + p_2L_2$ defined by

$$\begin{aligned}(p_1L_1 + p_2L_2)(y) &= p_1(x)L_1(y) + p_2(x)L_2(y) \\ &= [a_0(x) + p_2(x)b_0(x)]\, y^{(n)} + \cdots [p_1(x)a_n(x) + p_2(x)b_n(x)]\, y\end{aligned} \tag{3.3}$$

is again a linear differential operator. An important property of linear operators in general is the *distributive law*:

$$L(L_1 + L_2) = LL_1 + LL_2, \quad (L_1 + L_2)L = L_1L + L_2L.$$

The linearity of equation implies that for any two solutions y_1, y_2 the difference $y_1 - y_2$ is a solution of the associated homogeneous equation $L(y) = 0$. Moreover, it implies that any linear combination $a_1y_1 + a_2y_2$ of solutions y_1, y_2 of $L(y) = 0$ is again a solution of $L(y) = 0$. The solution space of $L(y) = 0$ is also called the **kernel** of L and is denoted by $\ker(L)$. It is a subspace of the vector space of real valued functions on some interval I. If y_p is a particular solution of $L(y) = b(x)$, the general solution of $L(y) = b(x)$ is

$$\ker(L) + y_p = \{y + y_p \mid L(y) = 0\}.$$

The differential operator $L(y) = y'$ may be denoted by D. The operator $L(y) = y''$ is nothing but $D^2 = D \circ D$ where $\circ$ denotes composition of functions. More generally, the operator $L(y) = y^{(n)}$ is D^n. The identity operator I is defined by $I(y) = y$. By definition $D^0 = I$. The general linear n-th order ODE can therefore be written

$$\Big[a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_n(x)I\Big](y) = b(x).$$

5. Basic Theory of Linear Differential Equations

In this lecture we will develop the theory of linear differential equations. The starting point is the fundamental existence theorem for the general n-th order ODE $L(y) = b(x)$, where

$$L(y) = D^n + a_1(x)D^{n-1} + \cdots + a_n(x).$$

We will also assume that $a_0(x), a_1(x), \ldots, a_n(x), b(x)$ are continuous functions on the interval I.

5.1 Basics of Linear Vector Space

5.1.1 Isomorphic Linear Transformation

From the fundamental theorem, it is known that for any $x_0 \in I$, the initial value problem

$$L(y) = b(x) \qquad y(x_0) = d_1, y'(x_0) = d_2, \ldots, y^{(n-1)}(x_0) = d_n$$

has a unique solution for any $d_1, d_2, \ldots, d_n \in \mathcal{R}$.

Thus, if V is the solution space of the associated homogeneous DE $L(y) = 0$, the transformation

$$T : V \to \mathcal{R}^n,$$

defined by $T(y) = (y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0))$, is linear transformation of the vector space V into $\mathcal{R}^n$ since

$$T(ay + bz) = aT(y) + bT(z).$$

Moreover, the fundamental theorem says that T is one-to-one ($T(y) = T(z)$ *impliesy* $= z$) and onto (every $d \in \mathcal{R}^n$ is of the form $T(y)$ for some $y \in V$). A linear transformation which is one-to-one and onto is called an **isomorphism**. Isomorphic vector spaces have the same properties.

5.1.2 Dimension and Basis of Vector Space

We call the vector space being n-dimensional with the notation by $\dim(V) = n$. This means that there exists a sequence of elements: $y_1, y_2, \ldots, y_n \in V$ such that every $y \in V$ can be uniquely written in the form

$$y = c_1 y_1 + c_2 y_2 + \ldots c_n y_n$$

with $c_1, c_2, \ldots, c_n \in \mathcal{R}$. Such a sequence of elements of a vector space V is called a **basis** for V. In the context of DE's it is also known as a **fundamental set**. The number of elements in a basis for V is called the dimension of V and is denoted by $\dim(V)$. If

$$e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$$

is the standard basis of $\mathcal{R}^n$ and y_i is the unique $y_i \in V$ with $T(y_i) = e_i$ then $y_1, y_2, \ldots, y_n$ is a basis for V. This follows from the fact that

$$T(c_1 y_1 + c_2 y_2 + \cdots + c_n y_n) = c_1 T(y_1) + c_2 T(y_2) + \cdots + c_n T(y_n).$$

5.1.3 (*) Span and Subspace

A set of vectors $v_1, v_2, \cdots, v_n$ in a vector space V is said to **span** or **generate** V if every $v \in V$ can be written in the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

with $c_1, c_2, \ldots, c_n \in \mathcal{R}$. Obviously, not any set of n vectors can span the vector space V. It will be seen that $\{v_1, v_2, \cdots, v_n\}$ span the vector space V, if and only if they are linear independent. The set

$$S = \operatorname{span}(v_1, v_2, \ldots, v_n) = \{c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \mid c_1, c_2, \ldots, c_n \in \mathcal{R}\}$$

consisting of all possible linear combinations of the vectors $v_1, v_2, \ldots, v_n$ form a **subspace** of V, which may be also called the **span** of $\{v_1, v_2, \ldots, v_n\}$. Then $V = \operatorname{span}(v_1, v_2, \ldots, v_n)$ if and only if $v_1, v_2, \ldots, v_n$ spans V.

5.1.4 Linear Independency

The vectors $v_1, v_2, \ldots, v_n$ are said to be **linearly independent** if

$$c_1 v_1 + c_2 v_2 + \ldots c_n v_n = 0$$

implies that the scalars $c_1, c_2, \ldots, c_n$ are all zero. A basis can also be characterized as a linearly independent generating set since the uniqueness of representation is equivalent to linear independence. More precisely,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = c_1' v_1 + c_2' v_2 + \cdots + c_n' v_n$$

implies

$$c_i = c_i' \qquad \text{for all } i,$$

if and only if $v_1, v_2, \ldots, v_n$ are linearly independent.

As an example of a linearly independent set of functions consider

$$\cos(x), \cos(2x), \sin(3x).$$

To prove their linear independence, suppose that c_1, c_2, c_3 are scalars such that

$$c_1 \cos(x) + c_2 \cos(2x) + c_3 \sin(3x) = 0$$

for all x. Then setting $x = 0, \pi/2, \pi$, we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ -c_2 - c_3 &= 0, \\ -c_1 + c_2 &= 0 \end{aligned} \tag{3.4}$$

from which $c_1 = c_2 = c_3 = 0$.

An example of a linearly dependent set would be $\sin^2(x), \cos^2(x), \cos(2x)$ since

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

implies that $\cos(2x) + \sin^2(x) + (-1)\cos^2(x) = 0$.

5.2 Wronskian of n-functions

Another criterion for linear independence of functions involves the Wronskian.

5.2.1 Definition

If $y_1, y_2, \ldots, y_n$ are n functions which have derivatives up to order $n-1$ then the Wronskian of these functions is the determinant

$$W = W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

If $W(x_0) \neq 0$ for some point x_0, then $y_1, y_2, \ldots, y_n$ are linearly independent. This follows from the fact that $W(x_0)$ is the determinant of the coefficient matrix of the linear homogeneous system of equations in $c_1, c_2, \ldots, c_n$ obtained from the dependence relation

$$c_1 y_1 + c_2 y_2 + \cdots + c_n y_n = 0$$

and its first $n-1$ derivatives by setting $x = x_0$.

For example, if $y_1 = \cos(x), \cos(2x), \cos(3x)$ we have

$$W = \begin{vmatrix} \cos(x) & \cos(2x) & \cos(3x) \\ -\sin(x) & -2\sin(2x) & -3\sin(3x) \\ -\cos(x) & -4\cos(2x) & -9\cos(3x) \end{vmatrix}$$

and $W(\pi/4)) = -8$ which implies that y_1, y_2, y_3 are linearly independent. Note that $W(0) = 0$ so that you cannot conclude linear dependence from the vanishing of the Wronskian at a point. This is not the case if $y_1, y_2, \ldots, y_n$ are solutions of an n-th order linear homogeneous ODE.

5.2.2 Theorem 1

The the Wronskian of n solutions of the n-th order linear ODE $L(y) = 0$ is subject to the following first order ODE:

$$\frac{dW}{dx} = -a_1(x)W,$$

with solution

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)dt}.$$

From the above it follows that the Wronskian of n solutions of the n-th order linear ODE $L(y) = 0$ is either identically zero or vanishes nowhere.

5.2.3 Theorem 2

If $y_1, y_2, \ldots, y_n$ are solutions of the linear ODE $L(y) = 0$, the following are equivalent:

1 $y_1, y_2, \ldots, y_n$ is a basis for the vector space $V = \ker(L)$;

2 $y_1, y_2, \ldots, y_n$ are linearly independent;

3 $^{(*)}$ $y_1, y_2, \ldots, y_n$ span V;

4 $y_1, y_2, \ldots, y_n$ generate $\ker(L)$;

5 $W(y_1, y_2, \ldots, y_n) \neq 0$ at some point x_0;

6 $W(y_1, y_2, \ldots, y_n)$ is never zero.

Proof. The equivalence of $1, 2, 3$ follows from the fact that $\ker(L)$ is isomorphic to $\mathcal{R}^n$. The rest of the proof follows from the fact that if the Wronskian were zero at some point x_0 the homogeneous system of equations

$$\begin{aligned}
&c_1y_1(x_0) + c_1y_2(x_0) + \cdots + c_ny_n(x_0) &&= 0\\
&c_1y_1'(x_0) + c_1y_2'(x_0) + \cdots + c_ny_n'(x_0) &&= 0\\
&\vdots\\
&c_1y_1^{(n-1)}(x_0) + c_1y_2^{(n-1)}(x_0) + \cdots + c_ny_n^{(n-1)}(x_0) &&= 0
\end{aligned} \tag{3.5}$$

would have a non-zero solution for $c_1, c_2, \ldots, c_n$ which would imply that

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

and hence that $y_1, y_2, \ldots, y_n$ are not linearly independent. **QED**

From the above, we see that to solve the n-th order linear DE $L(y) = b(x)$ we first find linear n independent solutions $y_1, y_2, \ldots, y_n$ of $L(y) = 0$. Then, if y_P is a particular solution of $L(y) = b(x)$, the general solution of $L(y) = b(x)$ is

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_P.$$

The initial conditions $y(x_0) = d_1, y'(x_0) = d_2, \ldots, y_n^{(n-1)}(x_0) = d_n$ then determine the constants $c_1, c_2, \ldots, c_n$ uniquely.

PART (III): SOLUTIONS FOR EQUATIONS WITH CONSTANTS COEFFICIENTS (1)

In what follows, we shall first focus on the linear equations with constant coefficients:

$$L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x)$$

and present two different approaches to solve them.

6. The Method with Undetermined Parameters

To illustrate the idea, as a special case, let us first consider the 2-nd order Linear equation with the constant coefficients:

$$L(y) = ay'' + by' + cy = f(x). \tag{3.6}$$

The associate homogeneous equation is:

$$L(y) = ay'' + by' + cy = 0. \tag{3.7}$$

6.1 Basic Equalities (I)

We first give the following basic identities:

$$D(\mathrm{e}^{rx}) = r\mathrm{e}^{rx};\ D^2(\mathrm{e}^{rx}) = r^2\mathrm{e}^{rx};\ \cdots\ D^n(\mathrm{e}^{rx}) = r^n\mathrm{e}^{rx}. \tag{3.8}$$

To solve this equation, we assume that the solution is in the form $y(x) = \mathrm{e}^{rx}$, where r is a constant to be determined. Due to the properties of the exponential function e^{rx}:

$$y'(x) = ry(x);\ y''(x) = r^2 y(x);\ \cdots\ y^{(n)} = r^n y(x),$$

we can write

$$L(\mathrm{e}^{rx}) = \phi(r)\mathrm{e}^{rx}. \tag{3.9}$$

for any given (r, x), where

$$\phi(r) = ar^2 + br + c.$$

is called the characteristic polynomial. From (3.9) it is seen that the function e^{rx} satisfies the equation (3.6), namely $L(\mathrm{e}^{rx}) = 0$, as long as the constant r is the root of the characteristic polynomial, i.e. $\phi(r) = 0$. In general, the polynomial $\phi(r)$ has two roots $(r1, r_2)$: One can write

$$\phi(r) = ar^2 + br + c = a(r - r_1)(r - r_2).$$

Accordingly, the equation (3.7) has two solutions:

$$\{y_1(x) = \mathrm{e}^{r_1 x};\ y_2(x) = \mathrm{e}^{r_2 x}\}.$$

Two cases should be discussed separately.

6.2 Cases (I) ($r_1 > r_2$)

When $b^2 - 4ac > 0$, the polynomial $\phi(r)$ has two distinct real roots $(r_1 \neq r_2)$. In this case, the two solutions, $y_(x); y_2(x)$ are different. The following linear combination is not only solution, but also the general solution of the equation:

$$y(x) = Ay_1(x) + By_2(x), \tag{3.10}$$

where A, B are arbitrary constants. To prove that, we make use of the fundamental theorem which states that if y, z are two solutions such that $y(0) = z(0) = y_0$ and $y'(0) = z'(0 = y_0')$ then $y = z$. Let y be any solution and consider the linear equations in A, B

$$\begin{aligned} Ay_1(0) + By_2(0) &= y(0), \\ Ay_1'(0) + By_2'(0) &= y'(0), \end{aligned} \tag{3.11}$$

or

$$\begin{aligned} A + B \quad &= y_0, \\ Ar_1 + Br_2 &= y_0'. \end{aligned} \tag{3.12}$$

Due to $r_1 \neq r_2$, these conditions leads to the unique solution A, B. With this choice of A, B the solution $z = Ay_1 + By_2$ satisfies $z(0) = y(0)$, $z'(0) = y'(0)$ and hence $y = z$. Thus, (3.10) contains all possible solutions of the equation, so, it is indeed the general solution.

6.3 Cases (II) ($r_1 = r_2$)

When $b^2 - 4ac = 0$, the polynomial $\phi(r)$ has double root: $r_1 = r_2 = \frac{-b}{2a}$. In this case, the solution $y_1(x) = y_2(x) = \mathrm{e}^{r_1 x}$. Thus, for the general solution, one needs to derive another type of the second solution. For this purpose, one may use the **method of reduction of order**.

Let us look for a solution of the form $C(x)e^{r_1 x}$ with the undetermined function $C(x)$. By substituting the equation, we derive that

$$L\Big(C(x)e^{r_1 x}\Big) = C(x)\phi(r_1)e^{r_1 x} + a\Big[C''(x) + 2r_1 C'(x)\Big]e^{r_1 x} + bC'(x)e^{r_1 x} = 0.$$

Noting that

$$\phi(r_1) = 0; \quad 2ar_1 + b = 0,$$

we get

$$C''(x) = 0$$

or

$$C(x) = Ax + B,$$

where A, B are arbitrary constants. Thus, we solution:

$$y(x) = (Ax + B)\mathrm{e}^{r_1 x}, \tag{3.13}$$

is a two parameter family of solutions consisting of the linear combinations of the two solutions $y_1 = e^{r_1 x}$ and $y_2 = xe^{r_1 x}$. It is also the general solution of the equation. The proof is similar to that given for the case (I) based on the fundamental theorem of existence and uniqueness. Let y be any solution and consider the linear equations in A, B

$$\begin{aligned} Ay_1(0) + By_2(0) &= y(0), \\ Ay_1'(0) + By_2'(0) &= y'(0), \end{aligned} \tag{3.14}$$

or

$$\begin{aligned} A \qquad &= y(0), \\ Ar_1 + B &= y'(0). \end{aligned} \tag{3.15}$$

these conditions leads to the unique solution $A = y(0), B = y'(0) - r_1 y(0)$. With this choice of A, B the solution $z = Ay_1 + By_2$ satisfies $z(0) = y(0)$, $z'(0) = y'(0)$ and hence $y = z$. Thus, (3.13) contains all possible solutions of the equation, so, it is indeed the general solution. The approach presented in this subsection is applicable to any higher order equations with constant coefficients.

Example 1. Consider the linear DE $y'' + 2y' + y = x$. Here $L(y) = y'' + 2y' + y$. A particular solution of the DE $L(y) = x$ is $y_p = x - 2$. The associated homogeneous equation is

$$y'' + 2y'' + y = 0.$$

The characteristic polynomial

$$\phi(r) = r^2 + 2r + 1 = (r+1)^2$$

has double roots $r_1 = r_2 = -1$. Thus the general solution of the DE

$$y'' + 2y' + y = x$$

is $y = Axe^{-x} + Be^{-x} + x - 2$.

This equation can be solved quite simply without the use of the fundamental theorem if we make essential use of operators.

6.4 Cases (III) ($r_{1,2} = \lambda \pm \mathrm{i}\mu$)

When $b^2 - 4ac < 0$, the polynomial $\phi(r)$ has two conjugate complex roots $r_{1,2} = \lambda \pm \mathrm{i}\mu$. We have to define the complex number,

$$\mathrm{i}^2 = -1; \qquad \mathrm{i}^3 = -\mathrm{i}; \qquad \mathrm{i}^4 = 1; \qquad \mathrm{i}^5 = \mathrm{i}, \cdots$$

and define and complex function with the Taylor series:

$$\begin{aligned} \mathrm{e}^{\mathrm{i}x} &= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} + \mathrm{i} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \cos x + \mathrm{i} \sin x. \end{aligned} \tag{3.16}$$

From the definition, it follows that

$$\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x (\cos y + \mathrm{i} \sin y).$$

and

$$D(\mathrm{e}^{rx}) = r\mathrm{e}^x, \qquad D^n(\mathrm{e}^{rx}) = r^n \mathrm{e}^x$$

where r is a complex number. So that, the basic equalities are now extended to the case with complex number r. Thus, we have the two complex solutions:

$$y_1(x) = \mathrm{e}^{r_1 x} = \mathrm{e}^{\lambda x}(\cos \mu x + \mathrm{i} \sin \mu x), \quad y_2(x) = \mathrm{e}^{r_2 x} = \mathrm{e}^{\lambda x}(\cos \mu x - \mathrm{i} \sin \mu x)$$

with a proper combination of these two solutions, one may derive two real solutions:

$$\tilde{y}_1(x) = \mathrm{e}^{\lambda x} \cos \mu x, \quad \tilde{y}_2(x) = \mathrm{e}^{\lambda x} \sin \mu x$$

and the general solution:

$$y(x) = \mathrm{e}^{\lambda x}(A \cos \mu x + B \sin \mu x).$$

PART (IV): SOLUTIONS FOR EQUATIONS WITH CONSTANTS COEFFICIENTS (2)

We adopt the differential operator D and write the linear equation in the following form:

$$L(y) = (a_0 D^{(n)} + a_1 D^{(n-1)} + \cdots + a_n)y = P(D)y = b(x).$$

7. The Method with Differential Operator

7.1 Basic Equalities (II).

We may prove the following basic identity of differential operators: for any scalar a,

$$\begin{aligned} (D-a) \; &= e^{ax} D e^{-ax} \\ (D-a)^n &= e^{ax} D^n e^{-ax} \end{aligned} \tag{3.17}$$

where the factors e^{ax}, e^{-ax} are interpreted as linear operators. This identity is just the fact that

$$\frac{dy}{dx} - ay = e^{ax}\left(\frac{d}{dx}(e^{-ax}y)\right).$$

The formula (3.17) may be extensively used in solving the type of linear equations under discussion. Let write the equation (3.7) with the differential operator in the following form:

$$L(y) = (aD^2 + bD + c)y = \phi(D)y = 0, \tag{3.18}$$

where

$$\phi(D) = (aD^2 + bD + c)$$

is a polynomial of D. We now re-consider the cases above-discussed with the previous method.

7.2 Cases (I) ($b^2 - 4ac > 0$)

The polynomial $\phi(r)$ have two distinct real roots $r_1 > r_2$. Then, we can factorize the polynomial $\phi(D) = (D - r_1)(D - r_2)$ and re-write the equation as:

$$L(y) = (D - r_1)(D - r_2)y = 0.$$

letting

$$z = (D - r_2)y,$$

in terms the basic equalities, we derive

$$(D - r_1)z = \mathrm{e}^{r_1 x} D\mathrm{e}^{-r_1 x} z = 0,$$

$$\mathrm{e}^{r_1 x} z = A, \quad z = A\mathrm{e}^{r_1 x}.$$

Furthermore, from

$$(D - r_2)y = \mathrm{e}^{r_2 x} D\mathrm{e}^{-r_2 x} y = z = A\mathrm{e}^{r_1 x},$$

we derive

$$D(\mathrm{e}^{-r_2 x} y) = z\mathrm{e}^{-r_2 x} = A\mathrm{e}^{(r_1 - r_2)x}$$

and

$$y = \tilde{A}\mathrm{e}^{r_1 x} + B\mathrm{e}^{r_2 x},$$

where $\tilde{A} = \frac{A}{(r_1 - r_2)}$, B are arbitrary constants. It is seen that, in general, to solve the equation

$$L(y) = (D - r_1)(D - r_2)\cdots(D - r_n)y = 0,$$

where $r_i \neq r_j, (i \neq j)$, one can first solve each factor equations

$$(D - r_i)y_i = 0, \qquad (i = 1, 2, \cdots, n)$$

separately. The general solution can be written in the form:

$$y(x) = y_1(x) + y_2(x) + \cdots + y_n(x).$$

7.3 Cases (II) ($b^2 - 4ac = 0$)

. The polynomial $\phi(r)$ have double real roots $r_1 = r_2$. Then, we can factorize the polynomial $\phi(D) = (D - r_1)^2$ and re-write the equation as:

$$L(y) = (D - r_1)^2 y = 0.$$

In terms the basic equalities, we derive

$$(D - r_1)^2 y = \mathrm{e}^{r_1 x} D^2 \mathrm{e}^{-r_1 x} y = 0,$$

hence,

$$D^2(\mathrm{e}^{-r_1 x}y) = 0.$$

One can solve

$$(\mathrm{e}^{-r_1 x}y) = A + Bx,$$

or

$$y = (A + Bx)\mathrm{e}^{r_1 x}.$$

In general, for the equation,

$$L(y) = (D - r_1)^n y = 0.$$

we have the general solution:

$$y = (A_1 + A_2 x + \cdots + A_n x^{n-1})\mathrm{e}^{r_1 x}.$$

So, we may write

$$\ker((D-a)^n) = \{(a_0 + a_x + \cdots + a_{n-1}x^{n-1})e^{ax} \mid a_0, a_1, \ldots, a_{n-1} \in \mathcal{R}\}.$$

7.4 Cases (III) ($b^2 - 4ac < 0$)

The polynomial $\phi(r)$ have two complex conjugate roots $r_{1,2} = \lambda \pm \mathrm{i}\mu$. Then, we can factorize the polynomial $\phi(D) = (D-\lambda)^2 + \mu^2$, and re-write the equation as:

$$L(y) = ((D-\lambda)^2 + \mu^2)y = 0. \tag{3.19}$$

Let us consider the special case first:

$$L(z) = (D^2 + \mu^2)z = 0.$$

From the formulas:

$$D(\cos \mu x) = -\mu \sin x, \qquad D(\sin x) = \mu \cos x,$$

it follows that

$$z(x) = A \cos \mu x + B \sin \mu x.$$

To solve for $y(x)$, we re-write the equation (3.19) as

$$(\mathrm{e}^{\lambda x} D^2 \mathrm{e}^{-\lambda x} + \mu^2)y = 0.$$

Then

$$D^2(\mathrm{e}^{-\lambda x}y) + \mu^2 \mathrm{e}^{-\lambda x}y = (D^2 + \mu^2)\mathrm{e}^{-\lambda x}y = 0.$$

Thus, we derive

$$\mathrm{e}^{-\lambda x}y(x) = A \cos \mu x + B \sin \mu x,$$

or

$$y(x) = \mathrm{e}^{\lambda x}(A\cos\mu x + B\sin\mu x). \tag{3.20}$$

One may also consider case (I) with the complex number r_1, r_2 and obtain the complex solution:

$$y(x) = \mathrm{e}^{\lambda x}(A\mathrm{e}^{\mathrm{i}\mu x} + B\mathrm{e}^{-\mathrm{i}\mu x}). \tag{3.21}$$

7.5 Theorems

In summary, it can be proved that the following results hold:

- $\ker((D-a)^m) = \operatorname{span}(e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax})$

  It means that $((D-a)^m)y = 0$ has a set of fundamental solutions:

  $$\left\{e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}\right\}$$

- $\ker((D-a)^2 + b^2)^m) = \operatorname{span}(e^{ax}f(x), xe^{ax}f(x), \ldots, x^{m-1}e^{ax}f(x))$, $f(x) = \cos(bx)$ or $\sin(bx)$

  It means that $((D-a)^2 + b^2)^m)y = 0$ has a set of fundamental solutions:

  $$\left\{e^{ax}f(x), xe^{ax}f(x), \ldots, x^{m-1}e^{ax}f(x)\right\},$$

  where $f(x) = \cos(bx)$ or $\sin(bx)$.

- $\ker(P(D)Q(D)) = \ker(P(D)) + \ker(Q(D)) = \{y_1 + y_2 \mid y_1 \in \ker(P(D)), y_2 \in \ker(Q(D))\}$, if $P(X), Q(X)$ are two polynomials with constant coefficients that have no common root. It means that if $P(X), Q(X)$ have no common roots, then the set of fundamental solutions for the operator $P(D)Q(D)$ is just the joint set:

  $$\{p_1(x), p_2(x), \cdots, p_n(x); q_1(x), q_2(x), \cdots q_m(x)\},$$

  where $\{p_1(x), p_2(x), \cdots, p_n(x)\}$ is the set of fundamental solutions for the operator $P(D)$, and $\{q_1(x), q_2(x), \cdots q_m(x)\}$ is the set of fundamental solutions for the operator $Q(D)$.

Example 1. By using the differential operation method, one can easily solve some inhomogeneous equations. For instance, let us reconsider the example 1. One may write the DE $y'' + 2y' + y = x$ in the operator form as

$$(D^2 + 2D + I)(y) = x.$$

The operator $(D^2 + 2D + I) = \phi(D)$ can be factored as $(D + I)^2$. With (3.17), we derive that

$$(D + I)^2 = (e^{-x}De^x)(e^{-x}De^x) = e^{-x}D^2e^x.$$

Consequently, the DE $(D + I)^2(y) = x$ can be written $e^{-x}D^2e^x(y) = x$ or

$$\frac{d^2}{dx}(e^x y) = xe^x$$

which on integrating twice gives

$$e^x y = xe^x - 2e^x + Ax + B, \quad y = x - 2 + Axe^{-x} + Be^{-x}.$$

We leave it to the reader to prove that

$$\ker((D - a)^n) = \{(a_0 + a_x + \cdots + a_{n-1}x^{n-1})e^{ax} \mid a_0, a_1, \ldots, a_{n-1} \in \mathcal{R}\}.$$

Example 2. Now consider the DE $y'' - 3y' + 2y = e^x$. In operator form this equation is

$$(D^2 - 3D + 2I)(y) = e^x.$$

Since $(D^2 - 3D + 2I) = (D - I)(D - 2I)$, this DE can be written

$$(D - I)(D - 2I)(y) = e^x.$$

Now let $z = (D - 2I)(y)$. Then $(D - I)(z) = e^x$, a first order linear DE which has the solution $z = xe^x + Ae^x$. Now $z = (D - 2I)(y)$ is the linear first order DE

$$y' - 2y = xe^x + Ae^x$$

which has the solution $y = e^x - xe^x - Ae^x + Be^{2x}$. Notice that $-Ae^x + Be^{2x}$ is the general solution of the associated homogeneous DE $y'' - 3y' + 2y = 0$ and that $e^x - xe^x$ is a particular solution of the original DE $y'' - 3y' + 2y = e^x$.

Example 3. Consider the DE

$$y'' + 2y' + 5y = \sin(x)$$

which in operator form is $(D^2 + 2D + 5I)(y) = \sin(x)$. Now

$$D^2 + 2D + 5I = (D + I)^2 + 4I$$

and so the associated homogeneous DE has the general solution

$$Ae^{-x}\cos(2x) + Be^{-x}\sin(2x).$$

All that remains is to find a particular solution of the original DE. We leave it to the reader to show that there is a particular solution of the form $C\cos(x) + D\sin(x)$.

Example 4. Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = 0, \quad y(0) = 1, y'(0) = y''(0) = 0.$$

The DE in operator form is $(D^3 - 3D^2 + 7D - 5)(y) = 0$. Since

$$\phi(r) = r^3 - 3r^2 + 7r - 5 = (r-1)(r^2 - 2r + 5) = (r-1)[(r-1)^2 + 4],$$

we have

$$\begin{aligned} L(y) &= (D^3 - 3D^2 + 7D - 5)(y) \\ &= (D-1)[(D-1)^2 + 4](y) \\ &= [(D-1)^2 + 4](D-1)(y) \\ &= 0. \end{aligned} \tag{3.22}$$

From here, it is seen that the solutions for

$$(D-1)(y) = 0, \tag{3.23}$$

namely,

$$y(x) = c_1 e^x, \tag{3.24}$$

and the solutions for

$$[(D-1)^2 + 4](y) = 0, \tag{3.25}$$

namely,

$$y(x) = c_2 e^x \cos(2x) + c_3 e^x \sin(2x), \tag{3.26}$$

must be the solutions for our equation (3.22). Thus, we derive that the following linear combination

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x), \tag{3.27}$$

must be the solutions for our equation (3.22). In solution (3.27), there are three arbitrary constants (c_1, c_2, c_3). One can prove that this solution is the general solution, which covers all possible solutions of (3.22). For instance, given the I.C.'s: $y(0) = 1, y'(0) = 0, y''(0) = 0$, from (3.27), we can derive

$$c_1 + c_2 = 1,$$

$$c_1 + c_2 + 2c_3 = 0,$$

$$c_1 - 3c_2 + 4c_3 = 0,$$

and find $c_1 = 5/4, c_2 = -1/4, c_3 = -1/2$.

PART (V): FINDING A PARTICULAR SOLUTION FOR INHOMOGENEOUS EQUATION

Variation of parameters is method for producing a particular solution of a special kind for the general linear DE in normal form

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$$

from a fundamental set $y_1, y_2, \ldots, y_n$ of solutions of the associated homogeneous equation.

8. The Differential Operator for Equations with Constant Coefficients

Given

$$L(y) = P(D)(y) = (a_0D^{(n)} + a_1D^{(n-1)} + \cdots + a_nD)y = b(x).$$

Assume that the inhomogeneous term $b(x)$ is a solution of linear equation:

$$Q(D)(b(x)) = 0.$$

Then we can transform the original inhomogeneous equation to the homogeneous equation by applying the differential operator $Q(D)$ to its both sides,

$$\Phi(D)(y) = Q(D)P(D)(y) = 0.$$

The operator $Q(D)$ is called the *Annihilator*. The above method is also called the *Annihilator Method*.

Example 1. Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = x + e^x, \quad y(0) = 1, y'(0) = y''(0) = 0.$$

This DE is non-homogeneous. The associated homogeneous equation was solved in the previous lecture. Note that in this example, In the inhomogeneous term $b(x) = x + e^x$ is in the kernel of $Q(D) = D^2(D-1)$. Hence, we have

$$D^2(D-1)^2((D-1)^2+4)(y) = 0$$

which yields $y = Ax + B + Cxe^x + c_1e^x + c_2e^x\cos(2x) + c_3e^x\sin(2x)$. This shows that there is a particular solution of the form $y_P = Ax + B + Cxe^x$ which is obtained by discarding the terms in the solution space of the associated homogeneous DE. Substituting this in the original DE we get

$$y''' - 3y'' + 7y' - 5y = 7A - 5B - 5Ax - Ce^x$$

which is equal to $x + e^x$ if and only if $7A - 5B = 0$, $-5A = 1$, $-C = 1$ so that $A = -1/5$, $B = -7/25$, $C = -1$. Hence the general solution is

$$y = c_1e^x + c_2e^x\cos(2x) + c_3e^x\sin(2x) - x/5 - 7/25 - xe^x.$$

To satisfy the initial condition $y(0) = 0, y'(0) = y''(0) = 0$ we must have

$$\begin{aligned} c_1 + c_2 \quad &= 32/25, \\ c_1 + c_2 + 2c_3 &= 6/5, \\ c_1 - 3c_2 + 4c_3 &= 2 \end{aligned} \tag{3.28}$$

which has the solution $c_1 = 3/2, c_2 = -11/50, c_3 = -1/25$.

It is evident that if the function $b(x)$ can not be eliminated by any linear operator $Q(D)$, the annihilator method will not applicable.

9. The Method of Variation of Parameters

In this method we try for a solution of the form

$$y_P = C_1(x)y_1 + C_2(x)y_2 + \cdots + C_n(x)y_n.$$

Then $y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n' + C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n$ and we impose the condition

$$C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n = 0.$$

Then $y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n'$ and hence

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'' + C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n'.$$

Again we impose the condition $C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' = 0$ so that

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'.$$

We do this for the first $n-1$ derivatives of y so that for $1 \leq k \leq n-1$

$$y_P^{(k)} = C_1(x)y_1^{(k)} + C_2(x)y_2^{(k)} + \cdots C_n(x)y_n^{(k)},$$

$$C_1'(x)y_1^{(k)} + C_2'(x)y_2^{(k)} + \cdots + C_n'(x)y_n^{(k)} = 0.$$

Now substituting $y_P, y_P', \ldots, y_P^{(n-1)}$ in $L(y) = b(x)$ we get

$$\begin{aligned} C_1(x)L(y_1) + C_2(x)L(y_2) + \cdots + C_n(x)L(y_n) + C_1'(x)y_1^{(n-1)} \\ +C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x). \end{aligned} \tag{3.29}$$

But $L(y_i) = 0$ for $1 \leq k \leq n$ so that

$$C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x).$$

We thus obtain the system of n linear equations for $C_1'(x), \ldots, C_n'(x)$

$$\begin{array}{ll} C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n & = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' & = 0, \\ \vdots & \\ C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} & = b(x). \end{array} \tag{3.30}$$

If we solve this system using Cramer's Rule and integrate, we find

$$C_i(x) = \int_{x_0}^{x} (-1)^{n+i} b(t) \frac{W_i}{W} dt$$

where $W = W(y_1, y_2, \ldots, y_n)$ and $W_i = W(y_1 \ldots, \hat{y}_i, \ldots, y_n)$ where the $\hat{}$ means that y_i is omitted. Note that the particular solution y_P found in this way satisfies

$$y_P(x_0) = y_P'(x_0) = \cdots = y_P^{(n-1)} = 0.$$

The point x_0 is any point in the interval of continuity of the $a_i(x)$ and $b(x)$. Note that y_P is a linear function of the function $b(x)$.

Example 2. Find the general solution of $y'' + y = 1/x$ on $x > 0$.

The general solution of $y'' + y = 0$ is $y = c_1 \cos(x) + c_2 \sin(x)$. Using variation of parameters with $y_1 = \cos(x)$, $y_2 = \sin(x)$, $b(x) = 1/x$ and $x_0 = 1$, we have $W = 1$, $W_1 = \sin(x)$, $W_2 = \cos(x)$ and we obtain the particular solution $y_p = C_1(x)\cos(x) + C_2(x)\sin(x)$ where

$$C_1(x) = -\int_1^x \frac{\sin(t)}{t} dt, \quad C_2(x) = \int_1^x \frac{\cos(t)}{t} dt.$$

The general solution of $y'' + y = 1/x$ on $x > 0$ is therefore

$$y = c_1 \cos(x) + c_2 \sin(x) - \left(\int_1^x \frac{\sin(t)}{t} dt \right) \cos(x) + \left(\int_1^x \frac{\cos(t)}{t} dt \right) \sin(x).$$

When applicable, the annihilator method is easier as one can see from the DE $y'' + y = e^x$. Here it is immediate that $y_p = e^x/2$ is a particular solution while variation of parameters gives

$$y_p = -\left(\int_0^x e^t \sin(t) dt \right) \cos(x) + \left(\int_0^x e^t \cos(t) dt \right) \sin(x).$$

The integrals can be evaluated using integration by parts:

$$\begin{aligned} \textstyle\int_0^x e^t \cos(t) dt &= e^x \cos(x) - 1 + \textstyle\int_0^x e^t \sin(t) dt \\ &= e^x \cos(x) + e^x \sin(x) - 1 - \textstyle\int_0^x e^t \cos(t) dt \end{aligned} \tag{3.31}$$

which gives

$$\int_0^x e^t \cos(t) dt = \left[e^x \cos(x) + e^x \sin(x) - 1 \right] / 2$$

$$\int_0^x e^t \sin(t) dt = e^x \sin(x) - \int_0^x e^t \cos(t) dt = \left[e^x \sin(x) - e^x \cos(x) + 1 \right] / 2$$

so that after simplification $y_p = e^x/2 - \cos(x)/2 - \sin(x)/2$.

PART (VI): LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS

In this lecture we will give a few techniques for solving certain linear differential equations with non-constant coefficients. We will mainly restrict our attention to second order equations. However, the techniques can be extended to higher order equations. The general second order linear DE is

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x).$$

This equation is called a non-constant coefficient equation if at least one of the functions p_i is not a constant function.

10. Euler Equations

An important example of a non-constant linear DE is Euler's equation

$$x^2y'' + axy' + by = 0,$$

where a, b are constants.

This equation has singularity at $x = 0$. The fundamental theorem of existence and uniqueness of solution holds in the region $x > 0$ and $x, 0$, respectively. So one must solve the problem in the region $x > 0$, or $x < 0$ separately. We first consider the region $x > 0$. This Euler equation can be transformed into a constant coefficient DE by the change of independent variable $x = \mathrm{e}^t$. This is most easily seen by noting that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = e^t\frac{dy}{dx} = xy'$$

so that $\frac{dy}{dx} = e^{-t}\frac{dy}{dt}$. In operator form, we have

$$\frac{d}{dt} = e^t\frac{d}{dx} = x\frac{d}{dx}.$$

If we set $D = \frac{d}{dt}$, we have $\frac{d}{dx} = e^{-t}D$ so that

$$\frac{d^2}{dx^2} = e^{-t}De^{-t}D = e^{-2t}e^{t}De^{-t}D = e^{-2t}(D-1)D$$

so that $x^2y'' = D(D-1)$. By induction one easily proves that

$$\frac{d^n}{dx^n} = e^{-nt}D(D-1)\cdots(D-n+1)$$

or $x^ny^{(n)} = D(D-1)\cdots(D-n+1)(y)$. With the variable t, Euler's equation becomes

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = q(e^t),$$

which is a linear constant coefficient DE. Solving this for y as a function of t and then making the change of variable $t = \ln(x)$, we obtain the solution of Euler's equation for y as a function of x.

For the region $x < 0$, we may let $-x = \mathrm{e}^t$, or $|x| = \mathrm{e}^t$. Then the equation

$$x^2y'' + axy' + by = 0, \quad (x < 0)$$

is changed to the same form

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0.$$

Hence, we have the solution $y(t) = y(\ln|x|)$ $(x < 0)$.

The above approach, can extend to solve the n-th order Euler equation

$$x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_ny = q(x),$$

where $a_1, a_2, \ldots a_n$ are constants.

Example 1. Solve $x^2y'' + xy' + y = \ln(x)$, $(x > 0)$.

Making the change of variable $x = e^t$ we obtain

$$\frac{d^2y}{dt^2} + y = t$$

whose general solution is $y = A\cos(t) + B\sin(t) + t$. Hence

$$y = A\cos(\ln(x)) + B\sin(\ln(x)) + \ln(x)$$

is the general solution of the given DE.

Example 2. Solve $x^3y''' + 2x^2y'' + xy' - y = 0, \quad (x > 0)$.

This is a third order Euler equation. Making the change of variable $x = e^t$, we get

$$\Big(D(D-1)(D-2) + 2D(D-1) + (D-1)\Big)(y) = (D-1)(D^2+1)(y) = 0$$

which has the general solution $y = c_1e^t + c_2\sin(t) + c_3\cos(t)$. Hence

$$y = c_1x + c_2\sin(\ln(x)) + c_3\cos(\ln(x))$$

is the general solution of the given DE.

11. Exact Equations

The DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$ is said to be exact if

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = \frac{d}{dx}(A(x)y' + B(x)y).$$

In this case the given DE is reduced to solving the linear DE

$$A(x)y' + B(x)y = \int q(x)dx + C$$

a linear first order DE. The exactness condition can be expressed in operator form as

$$p_0D^2 + p_1D + p_2 = D(AD + B).$$

Since $\frac{d}{dx}(A(x)y' + B(x)y) = A(x)y'' + (A'(x) + B(x))y' + B'(x)y$, the exactness condition holds if and only if $A(x), B(x)$ satisfy

$$A(x) = p_0(x), \quad B(x) = p_1(x) - p_0'(x), \quad B'(x) = p_2(x).$$

Since the last condition holds if and only if $p_1'(x) - p_0''(x) = p_2(x)$, we see that the given DE is exact if and only if

$$p_0'' - p_1' + p_2 = 0$$

in which case

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = \frac{d}{dx}(p_0(x)y' + (p_1(x) - p_0'(x))y).$$

Example 3. Solve the DE $xy'' + xy' + y = x, \quad (x > 0)$.

This is an exact equation since the given DE can be written

$$\frac{d}{dx}(xy' + (x-1)y) = x.$$

Integrating both sides, we get

$$xy' + (x-1)y = x^2/2 + A$$

which is a linear DE. The solution of this DE is left as an exercise.

12. Reduction of Order

If y_1 is a non-zero solution of a homogeneous linear n-th order DE, one can always find a second solution of the form $y = C(x)y_1$ where $C'(x)$ satisfies a homogeneous linear DE of order $n-1$. Since we can choose $C'(x) \neq 0$ we find in this way a second solution $y_2 = C(x)y_1$ which is not a scalar multiple of y_1. In particular for $n = 2$, we obtain a fundamental set of solutions y_1, y_2. Let us prove this for the second order DE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0.$$

If y_1 is a non-zero solution we try for a solution of the form $y = C(x)y_1$. Substituting $y = C(x)y_1$ in the above we get

$$p_0(x)\Big(C''(x)y_1 + 2C'(x)y_1' + C(x)y_1''\Big) + p_1(x)\Big(C'(x)y_1 + C(x)y_1'\Big) + p_2(x)C(x)y_1 = 0.$$

Simplifying, we get

$$p_0y_1C''(x) + (p_0y_1' + p_1y_1)C'(x) = 0$$

since $p_0y_1'' + p_1y_1' + p_2y_1 = 0$. This is a linear first order homogeneous DE for $C'(x)$. Note that to solve it we must work on an interval where $y_1(x) \neq 0$. However, the solution found can always be extended to the places where $y_1(x) = 0$ in a unique way by the fundamental theorem.

The above procedure can also be used to find a particular solution of the non-homogenous DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$ from a non-zero solution of $p_0(x)y'' + p_1(x)y' + p_2(x)y = 0$.

Example 4. Solve $y'' + xy' - y = 0$.

Here $y = x$ is a solution so we try for a solution of the form $y = C(x)x$. Substituting in the given DE, we get

$$C''(x)x + 2C'(x) + x(C'(x)x + C(x)) - C(x)x = 0$$

which simplifies to

$$xC''(x) + (x^2 + 2)C'(x) = 0.$$

Solving this linear DE for $C'(x)$, we get

$$C'(x) = Ae^{-x^2/2}/x^2$$

so that

$$C(x) = A\int \frac{dx}{x^2e^{x^2/2}} + B$$

Hence the general solution of the given DE is

$$y = A_1x + A_2x\int \frac{dx}{x^2e^{x^2/2}}.$$

Example 5. Solve $y'' + xy' - y = x^3e^x$.

By the previous example, the general solution of the associated homogeneous equation is

$$y_H = A_1x + A_2x\int \frac{dx}{x^2e^{x^2/2}}.$$

Substituting $y_p = xC(x)$ in the given DE we get

$$C''(x) + (x + 2/2)C'(x) = x^2e^x.$$

Solving for $C'(x)$ we obtain

$$C'(x) = \frac{1}{x^2e^{x^2/2}}\left(A_2 + \int x^4\mathrm{e}^{x+x^2/2}\mathrm{d}x\right) = A_2\frac{1}{x^2e^{x^2/2}} + H(x),$$

where

$$H(x) = \frac{1}{x^2e^{x^2/2}}\int x^4\mathrm{e}^{x+x^2/2}\mathrm{d}x.$$

This gives

$$C(x) = A_1 + A_2\int \frac{dx}{x^2e^{x^2/2}} + \int H(x)\mathrm{d}x,$$

We can therefore take

$$y_p = x\int H(x)\mathrm{d}x,$$

so that the general solution of the given DE is

$$y = A_1x + A_2x\int \frac{dx}{x^2e^{x^2/2}} + y_p(x) = y_H(x) + y_p(x).$$

PART (VII): SOME APPLICATIONS OF SECOND ORDER DE'S

13. (*) Vibration System

We now give an application of the theory of second order DE's to the description of the motion of a simple mass-spring mechanical system with a damping device. We assume that the damping force is proportional to the velocity of the mass. If there are no external forces we obtain the differential equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

where $x = x(t)$ is the displacement from equilibrium at time t of the mass of $m > 0$ units, $b \geq 0$ is the damping constant and $k > 0$ is the spring constant. In operator form with $D = \frac{d}{dt}$ this DE is, after normalizing,

$$\left(D^2 + \frac{b}{m}D + \frac{k}{m}\right)(x) = 0.$$

The characteristic polynomial $r^2 + (b/m)r + k/m$ has discriminant

$$\Delta = (b^2 - 4km)/m^2.$$

If $b^2 < 4km$ we have $\Delta < 0$ and the characteristic polynomial factorizes in the form $(r + b/2m)^2 + \omega^2$ with

$$\omega = \sqrt{4km - b^2}/2m = \sqrt{\frac{k}{m} - (b/2m)^2}.$$

In this case the characteristic polynomial has complex roots $-b/2m \pm i\omega$ and the general solution of the DE is

$$y = e^{-bt/2m}(A\cos(\omega t) + B\sin(\omega t) = Ce^{-bt/2m}\sin(\omega t + \theta)$$

where $C = \sqrt{A^2 + B^2}$ and $0 \leq \theta \leq 2\pi$ defined by $\cos(\theta) = A/C$, $\sin(\theta) = B/C$. The angle θ is called the **phase**. In this case we see that the mass oscillates with **frequency** $\omega/2\pi$ and decreasing amplitude. If $b = 0$ there is no damping and the mass oscillates with frequency $\omega/2\pi$ and constant amplitude; such motion is called **simple harmonic**.

If $b^2 \geq 4km$ we have $\Delta \geq 0$ and so the characteristic polynomial has real roots

$$r_1 = -b/2m + \sqrt{b^2 - 4km}/2m, \quad r_2 = -b/2m - \sqrt{b^2 - 4km}/2m$$

which are both negative. If $r_1 = r_2 = r$ the general solution of our DE is

$$y = Ae^{rt} + Bte^{rt}$$

and if $r_1 \neq r_2$ the general solution is

$$y = Ae^{r_1 t} + Be^{r_2 t}.$$

In both cases $y \to 0$ as $t \to \infty$. In the second case we have what is called **over damping** and in the first case the over damping is said to be **critical**. In each the mass returns to its equilibrium position without oscillations.

Suppose now that our mass-spring system is subject to an external force so that our DE now becomes

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t).$$

The function $F(t)$ is called the **forcing function** and measures the magnitude and direction of the external force. We consider the important special case where the forcing function is harmonic

$$F(f) = F_0 \cos(\gamma t), \quad F_0 > 0 \text{ a constant.}$$

We also assume that we have under-damping with damping constant $b > 0$. In this case the DE has a particular solution of the form

$$y_p = A_1 \cos(\gamma t) + A_2 \sin(\gamma t).$$

Substituting the the DE and simplifying, we get

$$((k-m\gamma^2)A_1 + b\gamma A_2)\cos(\gamma t) + (-b\gamma A_1 + (k-m\gamma^2)A_2)\sin(\gamma t) = F_0 \cos(\gamma t).$$

Setting the corresponding coefficients on both sides equal, we get

$$\begin{aligned} (k - m\gamma^2)A_1 + b\gamma A_2 &= F_0, \\ -b\gamma A_1 + (k - m\gamma^2)A_2 &= 0. \end{aligned} \tag{3.32}$$

Solving for A_1, A_2 we get

$$A_1 = \frac{F_0(k - m\gamma^2)}{(k - m\gamma^2)^2 + b^2\gamma^2}, \quad A_2 = \frac{F_0 b\gamma}{(k - m\gamma^2)^2 + b^2\gamma^2}$$

and

$$\begin{aligned} y_p &= \tfrac{F_0}{(k-m\gamma^2)^2+b^2\gamma^2}((k - m\gamma^2)\cos(\gamma t) + b\gamma\sin(\gamma t)) \\ &= \tfrac{F_0}{\sqrt{(k-m\gamma^2)^2+b^2\gamma^2}}\sin(\gamma t + \phi). \end{aligned} \tag{3.33}$$

The general solution of our DE is then

$$y = Ce^{-bt/2m}\sin(\omega t + \theta) + \frac{F_0}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}\sin(\gamma t + \phi).$$

Because of damping the first term tends to zero and is called the **transient** part of the solution. The second term, the **steady-state** part of the solution, is due to the presence of the forcing function $F_0\cos(\gamma t)$. It is harmonic with the same frequency $\gamma/2\pi$ but is out of phase with it by an angle $\phi - \pi/2$. The ratio of the magnitudes

$$M(\gamma) = \frac{1}{\sqrt{(k - m\gamma^2)^2 + b^2\gamma^2}}$$

is called the **gain** factor. The graph of the function $M(\gamma)$ is called the **resonance curve**. It has a maximum of

$$\frac{1}{b\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

when $\gamma = \gamma_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}$. The frequency $\gamma_r/2\pi$ is called the **resonance frequency** of the system. When $\gamma = \gamma_r$ the system is said to be in resonance with the external force. Note that $M(\gamma_r)$ gets arbitrarily large as $b \to 0$. We thus see that the system is subject to large oscillations if the damping constant is very small and the forcing function has a frequency near the resonance frequency of the system.

The above applies to a simple LRC electrical circuit where the differential equation for the current I is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + I/C = F(t)$$

where L is the inductance, R is the resistance and C is the capacitance. The resonance phenomenon is the reason why we can send and receive and amplify radio transmissions sent simultaneously but with different frequencies.

Chapter 4

SERIES SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

PART (I): SERIES SOLUTIONS NEAR A ORDINARY POINT

A function $f(x)$ of one variable x is said to be **analytic** at a point $x = x_0$ if it has a convergent power series expansion

$$f(x) = \sum_{0}^{\infty} a_n(x-x_0)^n = a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n+\cdots$$

for $|x - x_0| < R$, $R > 0$. This point $x = x_0$ is also called **ordinary point**. Otherwise, $f(x)$ is said to have a **singularity** at $x = x_0$. The largest such R (possibly $+\infty$) is called the **radius of convergence** of the power series. The series converges for every x with $|x - x_0| < R$ and diverges for every x with $|x - x_0| > R$. There is a formula for $R = \frac{1}{\ell}$, where

$$\ell = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}, \qquad \text{or} \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

if the latter limit exists. The same is true if $x,\ x_0,\ a_i$ are complex. For example,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

for $|x| < 1$. The radius of convergence of the series is 1. It is also equal to the distance from 0 to the nearest singularity $x = i$ of $1/(x^2 + 1)$ in the complex plane.

Power series can be integrated and differentiated within the interval (disk) of convergence. More precisely, for $|x - x_0| < R$ we have

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$\int_0^x \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n$$

and the resulting power series have R as radius of convergence. If $f(x)$, $g(x)$ are analytic at $x = x_0$ then so is $f(x)g(x)$ and $af+bg$ for any scalars a, b with radii of convergence at least that of the smaller of the radii of convergence the series for $f(x), g(x)$. If $f(x)$ is analytic at $x = x_0$ and $f(x_0) \neq 0$ then $1/f(x_0)$ is analytic at $x = x_0$ with radius of convergence equal to the distance from x_0 to the nearest zero of $f(x)$ in the complex plane.

The following theorem shows that linear DE's with analytic coefficients at x_0 have analytic solutions at x_0 with radius of convergence as big as the smallest of the radii of convergence of the coefficient functions.

1. Series Solutions near a Ordinary Point

1.1 Theorem

If $p_1(x), p_2(x), \ldots, p_n(x), q(x)$ are analytic at $x = x_0$, the solutions of the DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = q(x)$$

are analytic with radius of convergence $\geq$ the smallest of the radii of convergence of the coefficient functions $p_1(x), p_2(x), \ldots, p_n(x), q(x)$.

The proof of this result follows from the proof of fundamental existence and uniqueness theorem for linear DE's using elementary properties of analytic functions and the fact that uniform limits of analytic functions are analytic.

Example 1. The coefficients of the DE $y'' + y = 0$ are analytic everywhere, in particular at $x = 0$. Any solution $y = y(x)$ has therefore a series representation

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with infinite radius of convergence. We have

$$y' = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n, \quad y'' = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n.$$

Therefore, we have

$$y'' + y = \sum_{n=0}^{\infty} ((n+1)(n+2)a_{n+2} + a_n)x^n = 0$$

for all x. It follows that $(n+1)(n+2)a_{n+2} + a_n = 0$ for $n \geq 0$. Thus

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \quad \text{for } n \geq 0$$

from which we obtain

$$a_2 = -\frac{a_0}{1 \cdot 2}, \ a_3 = -\frac{a_1}{2 \cdot 3}, \ a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4},$$
$$a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}.$$

By induction one obtains

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}, \quad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$$

and hence that

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = a_0 \cos(x) + a_1 \sin(x).$$

Example 2. The simplest non-constant DE is $y'' + xy = 0$ which is known as Airy's equation. Its coefficients are analytic everywhere and so the solutions have a series representation

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with infinite radius of convergence. We have

$$\begin{aligned} y'' + xy &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^{n+1}, \\ &= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n, \qquad (4.1) \\ &= 2a_2 + \sum_{n=1}^{\infty} ((n+1)(n+2)a_{n+2} + a_{n-1})x^n = 0 \end{aligned}$$

from which we get $a_2 = 0$, $(n+1)(n+2)a_{n+2} + a_{n-1} = 0$ for $n \geq 1$. Since $a_2 = 0$ and

$$a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}, \quad \text{for } n \geq 1$$

we have

$$a_3 = -\frac{a_0}{2 \cdot 3}, \ a_4 = -\frac{a_1}{3 \cdot 4}, \ a_5 = 0,$$
$$a_6 = -\frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6},$$
$$a_7 = -\frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}.$$

By induction we get $a_{3n+2} = 0$ for $n \geq 0$ and

$$a_{3n} = (-1)^n \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n},$$

$$a_{3n+1} = (-1)^n \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n) \cdot (3n+1)}.$$

Hence $y = a_0 y_1 + a_1 y_2$ with

$$y_1 = 1 - \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} - \cdots + (-1)^n \frac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n} + \cdots,$$

$$y_2 = x - \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} - \cdots (-1)^n \frac{x^{3n+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n) \cdot (3n+1)} + \cdots.$$

For positive x the solutions of the DE $y'' + xy = 0$ behave like the solutions to a mass-spring system with variable spring constant. The solutions oscillate for $x > 0$ with increasing frequency as $|x| \to \infty$. For $x < 0$ the solutions are monotone. For example, y_1, y_2 are increasing functions of x for $x \leq 0$.

PART (II): SERIES SOLUTION NEAR A REGULAR SINGULAR POINT

In this lecture we investigate series solutions for the general linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x),$$

where the functions $a_1, a_2, \ldots, a_n, b$ are analytic at $x = x_0$. If $a_0(x_0) \neq 0$ the point $x = x_0$ is called an **ordinary point** of the DE. In this case, the solutions are analytic at $x = x_0$ since the normalized DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = q(x),$$

where $p_i(x) = a_i(x)/a_0(x), q(x) = b(x)/a_0(x)$, has coefficient functions which are analytic at $x = x_0$. If $a_0(x_0) = 0$, the point $x = x_0$ is said to be a **singular point** for the DE. If k is the multiplicity of the zero of $a_0(x)$ at $x = x_0$ and the multiplicities of the other coefficient functions at $x = x_0$ is as big then, on cancelling the common factor $(x - x_0)^k$ for $x \neq x_0$, the DE obtained holds even for $x = x_0$ by continuity, has analytic coefficient functions at $x = x_0$ and $x = x_0$ is an ordinary point. In this case the singularity is said to be **removable**. For example, the DE $xy'' + \sin(x)y' + xy = 0$ has a removable singularity at $x = 0$.

2. Series Solutions near a Regular Singular Point

In general, the solution of a linear DE in a neighborhood of a singularity is extremely difficult. However, there is an important special case where this can be done. For simplicity, we treat the case of the general second order homogeneous DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad (x > x_0),$$

with a singular point at $x = x_0$. Without loss of generality we can, after possibly a change of variable $x - x_0 = t$, assume that $x_0 = 0$. We say that $x = 0$ is a **regular singular point** if the normalized DE

$$y'' + p(x)y' + q(x)y = 0, \quad (x > 0),$$

is such that $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$. A necessary and sufficient condition for this is that

$$\lim_{x\to 0} xp(x) = p_0, \quad \lim_{x\to 0} x^2q(x) = q_0$$

exist and are finite. In this case

$$xp(x) = p_0 + p_1x + \cdots + p_nx^n + \cdots, \quad x^2q(x) = q_0 + q_1x + \cdots + q_nx^n + \cdots$$

and the given DE has the same solutions as the DE

$$x^2y'' + x(xp(x))y' + x^2q(x)y = 0.$$

This DE is an Euler DE if $xp(x) = p_0$, $x^2q(x) = q_0$. This suggests that we should look for solutions of the form

$$y = x^r\left(\sum_{n=0}^{\infty} a_nx^n\right) = \sum_{n=0}^{\infty} a_nx^{n+r},$$

with $a_0 \neq 0$. Substituting this in the DE gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + \left(\sum_{n=0}^{\infty} p_nx^n\right)\left(\sum_{n=0}^{\infty}(n+r)a_nx^{n+r}\right) + \left(\sum_{n=0}^{\infty} q_nx^n\right)\left(\sum_{n=0}^{\infty} a_nx^{n+r}\right) = 0$$

which, on expansion and simplification, becomes

$$a_0F(r)x^r + \sum_{n=1}^{\infty}\Big\{F(n+r)a_n + [(n+r-1)p_1 + q_1]a_{n-1} + \cdots + (rp_n + q_n)a_0\Big\}x^{n+r} = 0, \quad (4.2)$$

where $F(r) = r(r-1) + p_0r + q_0$. Equating coefficients to zero, we get

$$r(r-1) + p_0r + q_0 = 0, \quad (4.3)$$

the **indicial equation**, and

$$F(n+r)a_n = -[(n+r-1)p_1 + q_1]a_{n-1} - \cdots - (rp_n + q_n)a_0 \quad (4.4)$$

for $n \geq 1$. The indicial equation (4.3) has two roots: r_1, r_2. Three cases should be discussed separately.

2.1 Case (I): The roots $(r_1 - r_2 \neq N)$

Two roots do'nt differ by an integer. In this case, the above recursive equation (4.4) determines a_n uniquely for $r = r_1$ and $r = r_2$. If $a_n(r_i)$ is the solution for $r = r_i$ and $a_0 = 1$, we obtain the linearly independent solutions

$$y_1 = x^{r_1}\left(\sum_{n=0}^{\infty} a_n(r_1)x^n\right), \quad y_2 = x^{r_2}\left(\sum_{n=0}^{\infty} a_n(r_2)x^n\right).$$

It can be shown that the radius of convergence of the infinite series is the distance to the singularity of the DE nearest to the singularity $x = 0$. If $r_1 - r_2 = N \geq 0$, the above recursion equations can be solved for $r = r_1$ as above to give a solution

$$y_1 = x^{r_1}\left(\sum_{n=0}^{\infty} a_n(r_1)x^n\right).$$

A second linearly independent solution can then be found by reduction of order.

However, the series calculations can be quite involved and a simpler method exists which is based on solving the recursion equation for $a_n(r)$ as a ratio of polynomials of r. This can always be done since $F(n + r)$ is not the zero polynomial for any $n \geq 0$. If $a_n(r)$ is the solution with $a_0(r) = 1$ and we let

$$y = y(x, r) = x^r\left(\sum_{n=0}^{\infty} a_n(r)x^n\right). \tag{4.5}$$

Thus, we have the following equality with two variables (x, r):

$$x^2y'' + x^2p(x)y' + x^2q(x)y = a_0F(r)x^r = (r - r_1)(r - r_2)x^r. \tag{4.6}$$

2.2 Case (II): The roots $(r_1 = r_2)$

In this case, from the equality (4.6) we get

$$x^2y'' + x^2p(x)y' + x^2q(x)y = (r - r_1)^2x^r.$$

Differentiating this equation with respect to r, we get

$$x^2\left(\frac{\partial y}{\partial r}\right)'' + x^2p(x)\left(\frac{\partial y}{\partial r}\right)' + x^2q(x)\frac{\partial y}{\partial r} = 2(r - r_1) + (r - r_1)^2x^r\ln(x).$$

Setting $r = r_1$, we find that

$$y_2 = \frac{\partial y}{\partial r}(x, r_1) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n(r_1)x^n\right)\ln(x) + x^{r_1}\sum_{n=0}^{\infty} a_n'(r_1)x^n$$

$$= y_1 \ln(x) + x^{r_1} \sum_{n=0}^{\infty} a_n'(r_1) x^n,$$

where $a_n'(r)$ is the derivative of $a_n(r)$ with respect to r. This is a second linearly independent solution. Since this solution is unbounded as $x \to 0$, any solution of the given DE which is bounded as $x \to 0$ must be a scalar multiple of y_1.

2.3 Case (III): The roots ($r_1 - r_2 = N > 0$)

For this case, we let $z(x,r) = (r - r_2)y(x,r)$. Thus, from the equality (4.6) we get

$$x^2 z'' + x^2 p(x) z' + x^2 q(x) z = (r - r_1)(r - r_2)^2 x^r.$$

Differentiating this equation with respect to r, we get

$$x^2 \left(\frac{\partial z}{\partial r}\right)'' + x^2 p(x) \left(\frac{\partial z}{\partial r}\right)' + x^2 q(x) \frac{\partial z}{\partial r} = (r - r_1)(r - r_2)^2 x^r \ln(x)$$
$$+ (r - r_2)\Big[(r - r_2) + 2(r - r_1)\Big] x^r.$$

Setting $r = r_2$, we see that $y_2 = \frac{\partial z}{\partial r}(x, r_2)$ is a solution of the given DE. Letting $b_n(r) = (r - r_2)a_n(r)$, we have

$$\begin{aligned} F(n+r)b_n(r) &= -\Big[(n + r - 1)p_1 + q_1\Big] b_{n-1}(r) - \cdots \\ &\quad -(rp_n + q_n) b_0(r) \end{aligned} \tag{4.7}$$

and

$$y_2 = \lim_{r \to r_2} \left(x^r \ln(x) \sum_{n=0}^{\infty} b_n(r) x^n + x^r \sum_{n=0}^{\infty} b_n'(r) x^n \right). \tag{4.8}$$

Note that $a_n(r_2) \neq 0$, for $n = 1, 2, \ldots N - 1$. Hence, we have

$$b_0(r_2) = b_1(r_2) = b_2(r_2) = \cdots = b_{N-1}(r_2) = 0.$$

However, $a_N(r_2) = \infty$, as $F(r_2 + N) = F(r_1) = 0$. Hence, we have

$$b_N(r_2) = \lim_{r \to r_2} (r - r_2) a_n(r) = a < \infty,$$

subsequently,

$$\lim_{r \to r_2} x^r \ln(x) b_N(r) x^N = a x^{r_1} \ln(x).$$

Furthermore,

$$\begin{aligned} F(N + 1 + r_2) b_{N+1}(r_2) &= F(1 + r_1) b_{N+1}(r_2) \\ &= -(r_1 p_1 + q_1) b_N(r_2) - \cdots - (r_2 p_{N+1} + q_{N+1}) b_0(r_2) \\ &= -(r_1 p_1 + q_1) b_N(r_2) \end{aligned} \tag{4.9}$$

Thus,

$$b_{N+1}(r_2) = \frac{(r_1p_1+q_1)}{F(1+r_1)}b_N(r_2) = a\frac{(r_1p_1+q_1)}{F(1+r_1)} = aa_1(r_1). \quad (4.10)$$

Similarly, we have

$$\begin{aligned} F(N+2+r_2)b_{N+2}(r_2) &= F(2+r_1)b_{N+2}(r_2) \\ &= -[(1+r_1)p_1+q_1]b_{N+1}(r_2) - (r_1p_2+q_2)b_N(r_2) \\ &\quad - \cdots - (r_2p_{N+2}+q_{N+2})b_0(r_2) \\ &= -a[(1+r_1)p_1+q_1]a_1(r_1) - a(r_1p_2+q_2), \end{aligned} \quad (4.11)$$

then we obtain

$$b_{N+2}(r_2) = -a\frac{[(1+r_1)p_1+q_1]a_1(r_1)+(r_1p_2+q_2)}{F(2+r_1)} = aa_2(r_1). (4.12)$$

In general, we can write

$$b_{N+k}(r_2) = aa_k(r_1). \quad (4.13)$$

Substituting the above results to (4.8), we finally derive

$$\begin{aligned} y_2 &= ax^{r_1}\left(\sum_{n=0}^{\infty} a_n(r_1)x^n\right)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n'(r_2)x^n\right) \\ &= ay_1\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n'(r_2)x^n\right). \end{aligned} \quad (4.14)$$

This gives a second linearly independent solution.

The above method is due to Frobenius and is called the **Frobenius method**.

Example 1. The DE $2xy'' + y' + 2xy = 0$ has a regular singular point at $x = 0$ since $xp(x) = 1/2$ and $x^2q(x) = x^2$. The indicial equation is

$$r(r-1) + \frac{1}{2}r = r(r - \frac{1}{2}).$$

The roots are $r_1 = 1/2$, $r_2 = 0$ which do not differ by an integer. We have

$$\begin{aligned} (r+1)(r+\tfrac{1}{2})a_1 &= 0, \\ (n+r)(n+r-\tfrac{1}{2})a_n &= -a_{n-2} \quad \text{for } n \geq 2, \end{aligned} \quad (4.15)$$

so that $a_n = -2a_{n-2}/(r+n)(2r+2n-1)$ for $n \geq 2$. Hence $0 = a_1 = a_3 = \cdots a_{2n+1}$ for $n \geq 0$ and

$$\begin{aligned} a_2 &= -\frac{2}{(r+2)(2r+3)}a_0, \\ a_4 &= -\frac{2}{(r+4)(2r+7)}a_2 = \frac{2^2}{(r+2)(r+4)(2r+3)(2r+7)}a_0. \end{aligned}$$

It follows by induction that

$$\begin{aligned} a_{2n} &= (-1)^n \frac{2^n}{(r+2)(r+4)\cdots(r+2n)} \\ &\quad \times \frac{1}{(2r+3)(2r+4)\cdots(2r+2n-1)}a_0. \end{aligned} \tag{4.16}$$

Setting, $r = 1/2,\ 0,\ a_0 = 1$, we get

$$y_1 = \sqrt{x}\sum_{n=0}^{\infty} \frac{x^{2n}}{(5\cdot 9\cdots(4n+1))n!}, \quad y_2 = \sum_{n=0}^{\infty} \frac{x^{2n}}{(3\cdot 7\cdot\cdots\cdot(4n-1))n!}.$$

The infinite series have an infinite radius of convergence since $x = 0$ is the only singular point of the DE.

Example 2. The DE $xy'' + y' + y = 0$ has a regular singular point at $x = 0$ with $xp(x) = 1$, $x^2 q(x) = x$. The indicial equation is

$$r(r-1) + r = r^2 = 0.$$

This equation has only one root $x = 0$. The recursion equation is

$$(n+r)^2 a_n = -a_{n-1}, \quad n \geq 1.$$

The solution with $a_0 = 1$ is

$$a_n(r) = (-1)^n \frac{1}{(r+1)^2(r+2)^2\cdots(r+n)^2}.$$

setting $r = 0$ gives the solution

$$y_1 = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{(n!)^2}.$$

Taking the derivative of $a_n(r)$ with respect to r we get, using

$$a_n'(r) = a_n(r)\frac{d}{dr}\ln\left[a_n(r)\right]$$

(logarithmic differentiation),we get

$$a_n'(r) = -\left(\frac{2}{r+1} + \frac{2}{r+2} + \cdots + \frac{2}{r+n}\right) a_n(r)$$

so that

$$a_n'(0) = 2(-1)^n \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{(n!)^2}.$$

Therefore a second linearly independent solution is

$$y_2 = y_1 \ln(x) + 2\sum_{n=1}^{\infty} (-1)^n \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{(n!)^2} x^n.$$

The above series converge for all x. Any bounded solution of the given DE must be a scalar multiple of y_1.

PART (III): BESSEL FUNCTIONS

3. Bessel Equation

In this lecture we study an important class of functions which are defined by the differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

where $\nu \geq 0$ is a fixed parameter. This DE is known **Bessel's equation of order** ν. This equation has $x = 0$ as its only singular point. Moreover, this singular point is a regular singular point since

$$xp(x) = 1, \quad x^2q(x) = x^2 - \nu^2.$$

Bessel's equation can also be written

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2}) = 0$$

which for x large is approximately the DE $y'' + y = 0$ so that we can expect the solutions to oscillate for x large. The indicial equation is $r(r-1) + r - \nu^2 = r - \nu^2$ whose roots are $r_1 = \nu$ and $r_2 = -\nu$. The recursion equations are

$$[(1+r)^2 - \nu^2]a_1 = 0, \quad [(n+r)^2 - \nu^2]a_n = -a_{n-2}, \quad \text{for } n \geq 2.$$

The general solution of these equations is $a_{2n+1} = 0$ for $n \geq 0$ and

$$\begin{aligned} a_{2n}(r) = & \frac{(-1)^n a_0}{(r+2-\nu)(r+4-\nu)\cdots(r+2n-\nu)} \\ & \times \frac{1}{(r+2+\nu)(r+4+\nu)\cdots(r+2n+\nu)}. \end{aligned}$$

4. The Case of Non-integer ν

If ν is not an integer and $\nu \neq 1/2$, we have the case (I). Two linearly independent solutions of Bessel's equation $J_\nu(x)$, $J_{-\nu}(x)$ can be obtained by taking $r = \pm\nu$, $a_0 = 1/2^\nu\Gamma(\nu+1)$. Since, in this case,

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(r+1)(r+2)\cdots(r+n)},$$

we have for $r = \pm\nu$

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+r}.$$

Recall that the Gamma function $\Gamma(x)$ is defined for $x \geq -1$ by

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x dt.$$

For $x \geq 0$ we have $\Gamma(x+1) = x\Gamma(x)$, so that $\Gamma(n+1) = n!$ for n an integer ≥ 0. We have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} dt = 2\int_0^\infty e^{-x^2} dt = \sqrt{\pi}.$$

The Gamma function can be extended uniquely for all x except for $x = 0, -1, -2, \ldots, -n, \ldots$ to a function which satisfies the identity $\Gamma(x) = \Gamma(x)/x$. This is true even if x is taken to be complex. The resulting function is analytic except at zero and the negative integers where it has a simple pole.

These functions are called **Bessel functions of first kind of order ν.**

As an exercise the reader can show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x), \quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cos(x).$$

5. The Case of $\nu = -m$ with m an integer ≥ 0

For this case, the first solution $J_m(x)$ can be obtained as in the last section. As examples, we give some such solutions as follows:

- The Case of $m = 0$:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

- The case $m = 1$:

$$J_1(x) = \frac{1}{2}y_1(x) = \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}n!(n+1)!}x^{2n}.$$

To derive the second solution, one has to proceed differently. For $\nu = 0$ the indicial equation has a repeated root, we have the case (II). One has a second solution of the form

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty} a'_{2n}(0)x^{2n}$$

where

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2\cdots(r+2n)^2}.$$

It follows that

$$\frac{a'_{2n}(r)}{a_{2n}} = -2\left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots \frac{1}{r+2n}\right)$$

so that

$$a'_{2n}(0) = -\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)a_{2n}(0) = -h_n a_{2n}(0),$$

where we have defined

$$h_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Hence

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}h_n}{2^{2n}(n!)^2}x^{2n}.$$

Instead of y_2, the second solution is usually taken to be a certain linear combination of y_2 and J_0. For example, the function

$$Y_0(x) = \frac{2}{\pi}\Big[y_2(x) + (\gamma - \ln 2)J_0(x)\Big],$$

where $\gamma = \lim_{n\to\infty}(h_n - \ln n) \approx 0.5772$, is known as the **Weber function of order** 0. The constant γ is known as Euler's constant; it is not known whether γ is rational or not.

If $\nu = -m$, with $m > 0$, the the roots of the indicial equation differ by an integer, we have the case (III). Then one has a solution of the form

$$y_2 = aJ_m(x)\ln(x) + \sum_{n=0}^{\infty} b'_{2n}(-m)x^{2n+m}$$

where $b_{2n}(r) = (r+m)a_{2n}(r)$ and $a = b_{2m}(-m)$. In the case $m = 1$ we have $a_0 = 1$,

$$a = b_2(-1) = -\frac{a_0}{2},$$

$$b_0(r) = (r - r_2)a_0$$

and for $n \geq 1$,

$$b_{2n}(r) = \frac{(-1)^n a_0}{(r+3)(r+5)\cdots(r+2n-1)(r+3)(r+5)\cdots(r+2n+1)}.$$

Subsequently, we have

$$b_0'(r) = a_0$$

and for $n \geq 1$,

$$b_{2n}'(r) = -\left(\frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n-1} + \frac{1}{r+3} + \frac{1}{r+5} + \cdots + \frac{1}{r+2n+1}\right) b_{2n}(r).$$

From here, we obtain

$$\begin{aligned} b_0'(-1) &= a_0 \\ b_{2n}'(-1) &= \tfrac{-1}{2}(h_n + h_{n-1})b_{2n}(-1) \qquad (n \geq 1), \end{aligned} \tag{4.17}$$

where

$$b_{2n}(-1) = \frac{(-1)^n}{2^{2n}(n-1)!n!}a_0.$$

So that

$$\begin{aligned} y_2 &= \frac{-1}{2} y_1(x)\ln(x) + \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n+1}(n-1)!n!} x^{2n}\right] \\ &= -J_1(x)\ln(x) + \frac{1}{x}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n + h_{n-1})}{2^{2n+1}(n-1)!n!} x^{2n}\right] \end{aligned}$$

where, by convention, $h_0 = 0$, $(-1)! = 1$. The **Weber function of order 1** is defined to be

$$Y_1(x) = \frac{4}{\pi}\Big[-y_2(x) + (\gamma - \ln 2)J_1(x)\Big].$$

The case $m > 1$ is slightly more complicated and will not be treated here.

The second solutions $y_2(x)$ of Bessel's equation of order $m \geq 0$ are unbounded as $x \to 0$. It follows that any solution of Bessel's equation of order $m \geq 0$ which is bounded as $x \to 0$ is a scalar multiple of J_m. The solutions which are unbounded as $x \to 0$ are called **Bessel functions of the second kind**. The Weber functions are Bessel functions of the second kind.

Chapter 5

LAPLACE TRANSFORMS

PART (I): LAPLACE TRANSFORM AND ITS INVERSE

1. Introduction

We begin our study of the Laplace Transform with a motivating example. This example illustrates the type of problem that the Laplace transform was designed to solve.

A mass-spring system consisting of a single steel ball is suspended from the ceiling by a spring. For simplicity, we assume that the mass and spring constant are 1. Below the ball we introduce an electromagnet controlled by a switch. Assume that, we on, the electromagnet exerts a unit force on the ball. After the ball is in equilibrium for 10 seconds the electromagnet is turned on for 2π seconds and then turned off. Let $y = y(t)$ be the downward displacement of the ball from the equilibrium position at time t. To describe the motion of the ball using techniques previously developed we have to divide the problem into three parts: (I) $0 \le t < 10$; (II) $10 \le t < 10 + 2\pi$; (III) $10 + 2\pi \le t$. The initial value problem determining the motion in part I is

$$y'' + y = 0, \quad y(0) = y'(0) = 0.$$

The solution is $y(t) = 0,\ 0 \le t < 10$. Taking limits as $t \to 10$ from the left, we find $y(10) = y'(10) = 0$. The initial value problem determining the motion in part II is

$$y'' + y = 1, \quad y(10) = y'(10) = 0.$$

The solution is $y(t) = 1 - \cos(t - 10),\ 10 \le t < 2\pi + 10$. Taking limits as $t \to 10 + 2\pi$ from the left, we get $y(10 + 2\pi) = y'(10 + 2\pi) = 0$. The initial value problem for the last part is

$$y'' + y = 0, \quad y(10 + 2\pi) = y'(10 + 2\pi) = 0$$

which has the solution $y(t) = 0,\ 10 + 2\pi \le t$. Putting all this together, we have

$$y(t) = \begin{cases} 0, & 0 \le t < 10, \\ 1 - \cos(t - 10), & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

The function $y(t)$ is continuous with continuous derivative

$$y'(t) = \begin{cases} 0, & 0 \le t < 10, \\ \sin(t - 10), & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

However the function $y'(t)$ is not differentiable at $t = 10$ and $t = 10+2\pi$. In fact

$$y''(t) = \begin{cases} 0, & 0 \le t < 10, \\ \cos(t - 10), & 10 < t < 10 + 2\pi, \\ 0, & 10 + 2\pi < t. \end{cases}$$

The left hand and right hand limits of $f''(t)$ at $t = 10$ are 0 and 1 respectively. At $t = 10+2\pi$ they are 1 and 0. If we extend $y''(t)$ by using the left-hand or righthand limits at 10 and $10+2\pi$ the resulting function is not continuous. Such a function with only jump discontinuities is said to be **piecewise continuous**. If we try to write the differential equation of the system we have

$$y'' + y = f(t) = \begin{cases} 0, & 0 \le t < 10, \\ 1, & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

Here $f(t)$ is piecewise continuous and any solution would also have y'' piecewise continuous. By a solution we mean any function $y = y(t)$ satisfying the DE for those t not equal to the points of discontinuity of $f(t)$. In this case we have shown that a solution exists with $y(t)$, $y'(t)$ continuous. In the same way, one can show that in general such solutions exist using the fundamental theorem.

What we want to describe now is a mechanism for doing such problems without having to divide the problem into parts. This mechanism is the Laplace transform.

2. Laplace Transform

2.1 Definition

Let $f(t)$ be a function defined for $t \ge 0$. The function $f(t)$ is said to be **piecewise continuous** if

(1) $f(t)$ converges to a finite limit $f(0_+)$ as $t \to 0_+$

(2) for any $c > 0$, the left and right hand limits $f(c_-)$, $f(c_+)$ of $f(t)$ at c exist and are finite.
(3) $f(c_-) = f(c_+) = f(c)$ for every $c > 0$ except possibly a finite set of points or an infinite sequence of points converging to $+\infty$. Thus the only points of discontinuity of $f(t)$ are jump discontinuities. The function is said to be **normalized** if $f(c) = f(c_+)$ for every $c \geq 0$.

The Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is the function of a new variable s defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{N\to+\infty} \int_0^N e^{-st} f(t)dt.$$

An important class of functions for which the integral converges are the functions of exponential order. The function $f(t)$ is said to be of **exponential order** if there are constants a, M such that

$$|f(t)| \leq Me^{at}$$

for all t. the solutions of constant coefficient homogeneous DE's are all of exponential order. The convergence of the improper integral follows from

$$\int_0^N |e^{-st} f(t)|dt \leq M \int_0^N e^{-(s-a)t} dt = \frac{1}{s-a} - \frac{e^{-(s-a)}}{s-a}$$

which shows that the improper integral converges absolutely when $s > a$. It shows that $F(s) \to 0$ as $s \to \infty$. The calculation also shows that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

for $s > a$. Setting $a = 0$, we get $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

2.2 Basic Properties and Formulas

The above holds when $f(t)$ is complex valued and $s = \sigma + i\tau$ is complex. The integral exists in this case for $\sigma > a$. For example, this yields

$$\mathcal{L}\{e^{it}\} = \frac{1}{s-i}, \quad \mathcal{L}\{e^{-it}\} = \frac{1}{s+i}.$$

2.2.1 Linearity of the transform

$$\mathcal{L}\{af(t) + bf(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

Using this linearity property of the Laplace transform and using $\sin(t) = (e^{it} - e^{-it})/2i$, $\cos(t) = (e^{it} + e^{-it})/2$, we find

$$\mathcal{L}\{\sin(bt)\} = \frac{1}{2i}\left(\frac{1}{s-bi} - \frac{1}{s+bi}\right) = \frac{b}{s^2+b^2},$$

$$\mathcal{L}\{\cos(bt)\} = \frac{1}{2}\left(\frac{1}{s-bi} + \frac{1}{s+bi}\right) = \frac{s}{s^2+b^2},$$

for $s > 0$.

2.2.2 Formula (I)

The following two identities follow from the definition of the Laplace transform after a change of variable.

$$\begin{aligned} \mathcal{L}\{e^{at}f(t)\}(s) &= \mathcal{L}\{f(t)\}(s-a), \\ \mathcal{L}\{f(bt)\}(s) &= \frac{1}{b}\mathcal{L}\{f(t)\}(s/b). \end{aligned}$$

Using the first of these formulas, we get

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \qquad \mathcal{L}\{e^{at}\cos(t)\} = \frac{s-a}{(s-a)^2+b^2}.$$

2.2.3 Formula (II)

The next formula will allow us to find the Laplace transform for all the functions that are annihilated by a constant coefficient differential operator.

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s).$$

For $n = 1$ this follows from the definition of the Laplace transform on differentiating with respect s and taking the derivative inside the integral. The general case follows by induction. For example, using this formula, we obtain using $f(t) = 1$

$$\mathcal{L}\{t^n\}(s) = -\frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}}.$$

With $f(t) = \sin(t)$ and $f(t) = \cos(t)$ we get

$$\mathcal{L}\{t\sin(bt)\}(s) = -\frac{d}{ds}\frac{b}{s^2+b^2} = \frac{2bs}{(s^2+b^2)^2},$$

$$\mathcal{L}\{t\cos(bt)\}(s) = -\frac{d}{ds}\frac{s}{s^2+b^2} = \frac{s^2-b^2}{(s^2+b^2)^2} = \frac{1}{s^2+b^2} - \frac{2b^2}{(s^2+b^2)^2}.$$

2.2.4 Formula (III)

The next formula shows how to compute the Laplace transform of $f'(t)$ in terms of the Laplace transform of $f(t)$.

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

This follows from

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)dt = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$
$$= s\int_0^\infty e^{-st}f(t)dt - f(0) \tag{5.1}$$

since $e^{-st}f(t)$ converges to 0 as $t \to +\infty$ in the domain of definition of the Laplace transform of $f(t)$. To ensure that the first integral is defined, we have to assume $f'(t)$ is piecewise continuous. Repeated applications of this formula give

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0).$$

The following theorem is important for the application of the Laplace transform to differential equations.

3. Inverse Laplace Transform

3.1 Theorem:

If $f(t)$, $g(t)$ are normalized piecewise continuous functions of exponential order then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$$

implies

$$f = g.$$

3.2 Definition

If $F(s)$ is the Laplace of the normalized piecewise continuous function $f(t)$ of exponential order then $f(t)$ is called the **inverse Laplace transform** of $F(s)$. This is denoted by

$$F(s) = \mathcal{L}^{-1}\{f(t)\}.$$

Note that the inverse Laplace transform is also linear. Using the Laplace transforms we found for $t\sin(bt)$, $t\cos(bt)$ we find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{1}{2b}t\sin(bt),$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^2}\right\} = \frac{1}{2b^3}\sin(bt) - \frac{1}{2b^2}t\cos(bt).$$

PART (II): SOLVE DE'S WITH LAPLACE TRANSFORMS

4. Solve IVP of DE's with Laplace Transform Method

In this lecture we will, by using examples, show how to use Laplace transforms in solving differential equations with constant coefficients.

4.1 Example 1

Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

Step 1

Let $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\begin{aligned}\mathcal{L}\{y'(t)\} &= sY(s) - y(0) = sY(s) - 1,\\ \mathcal{L}\{y''(t)\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1.\end{aligned}$$

Taking Laplace transforms of the DE, we get

$$(s^2 + s + 1)Y(s) - s = \frac{1}{s^2+1}.$$

Step 2

Solving for $Y(s)$, we get

$$Y(s) = \frac{s}{s^2+s+1} + \frac{1}{(s^2+s+1)(s^2+1)}.$$

Step 3

Finding the inverse laplace transform.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\}.$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+1/2)^2+3/4} = \frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-t/2}\cos(\sqrt{3}\ t/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2).$$

Using partial fractions we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

Multiplying both sides by $(s^2+1)(s^2+s+1)$ and collecting terms, we find

$$1 = (A+C)s^3 + (B+C+D)s^2 + (A+C+D)s + B + D.$$

Equating coefficients, we get $A+C=0$, $B+C+D=0$, $A+C+D=0$, $B+D=1$,

from which we get $A=B=1$, $C=-1$, $D=0$, so that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2), \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

we obtain

$$y(t) = 2e^{-t/2}\cos(\sqrt{3}\ t/2) - \cos(t).$$

4.2 Example 2

As we have known, a higher order DE can be reduced to a system of DE's. Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= -2x + y, \\ \frac{dy}{dt} &= x - 2y \end{aligned} \tag{5.2}$$

with the initial conditions $x(0) = 1$, $y(0) = 2$.

Step 1

Taking Laplace transforms the system becomes

$$\begin{aligned} sX(s) - 1 &= -2X(s) + Y(s), \\ sY(s) - 2 &= X(s) - 2Y(s), \end{aligned} \tag{5.3}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$.

Step 2

Solving for $X(s)$, $Y(s)$. The above linear system of equations can be written in the form:

$$\begin{aligned} (s+2)X(s) - Y(s) &= 1, \\ -X(s) + (s+2)Y(s) &= 2. \end{aligned} \tag{5.4}$$

The determinant of the coefficient matrix is $s^2 + 4s + 3 = (s+1)(s+3)$. Using Cramer's rule we get

$$X(s) = \frac{s+4}{s^2+4s+3}, \quad Y(s) = \frac{2s+5}{s^2+4s+3}.$$

Step 3

Finding the inverse Laplace transform. Since

$$\frac{s+4}{(s+1)(s+3)} = \frac{3/2}{s+1} - \frac{1/2}{s+3}, \quad \frac{2s+5}{(s+1)(s+3)} = \frac{3/2}{s+1} + \frac{1/2}{s+3},$$

we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

The Laplace transform reduces the solution of differential equations to a partial fractions calculation. If $F(s) = P(s)/Q(s)$ is a ratio of polynomials with the degree of $P(s)$ less than the degree of $Q(s)$ then $F(s)$ can be written as a sum of terms each of which corresponds to an irreducible factor of $Q(s)$. Each factor $Q(s)$ of the form $s-a$ contributes the terms

$$\frac{A_1}{s-a} + \frac{A_1}{(s-a)^2} + \cdots + \frac{A_r}{(s-a)^r}$$

where r is the multiplicity of the factor $s-a$. Each irreducible quadratic factor $s^2 + as + b$ contributes the terms

$$\frac{A_1 s + B_1}{s^2 + as + b} + \frac{A_2 s + B_2}{(s^2 + as + b)^2} + \cdots + \frac{A_r s + B_r}{(s^2 + as + b)^r}$$

where r is the degree of multiplicity of the factor $s^2 + as + b$.

PART (III): FURTHER STUDIES OF LAPLACE TRANSFORM

5. Step Function

5.1 Definition

$$u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$$

5.2 Laplace transform of unit step function

$$\mathcal{L}\{u_c(t)\} = \frac{\mathrm{e}^{-cs}}{s}.$$

One can derive

$$\mathcal{L}\{u_c(t)f(t-c)\} = \mathrm{e}^{-cs}F(s).$$

6. Impulse Function

6.1 Definition

Let

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & |t| < \tau, \\ 0 & |t| \geq \tau. \end{cases}$$

It follows that

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\mathrm{d}t = 1.$$

Now, consider the limit,

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t) = \begin{cases} 0 & t \neq 0, \\ \infty & t = 0, \end{cases}$$

which is called the Dirac δ-function. Evidently, the Dirac δ-function has the following properties:

1
$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$$

2
$$\int_A^B \delta(t-c)\mathrm{d}t = \begin{cases} 0 & c \,\bar{\in} (A,B), \\ 1 & c \in (A,B). \end{cases}$$

3
$$\int_A^B \delta(t-c)f(t)\mathrm{d}t = \begin{cases} 0 & c \,\bar{\in} (A,B), \\ f(c) & c \in (A,B). \end{cases}$$

6.2 Laplace transform of unit step function

$$\mathcal{L}\{\delta(t-c)\} = \begin{cases} \mathrm{e}^{-cs} & c > 0, \\ 0 & c < 0. \end{cases}$$

One can derive

$$\mathcal{L}\{\delta(t-c)f(t)\} = \mathrm{e}^{-cs}f(c), \quad (c > 0).$$

7. Convolution Integral

7.1 Theorem

Given
$$\mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s),$$
one can derive
$$\mathcal{L}^{-1}\{F(s) \cdot G(s)\} = f(t) * g(t),$$
where
$$f(t) * g(t) = \int_0^{\tau} f(t-\tau)g(\tau)\mathrm{d}\tau$$
is called the convolution integral.

Chapter 6

(*) SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

(*) PART (I): INTRODUCTION OF SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

In this and the following lecture we will give an introduction to systems of differential equations. For simplicity, we will limit ourselves to systems of two equations with two unknowns. The techniques introduced can be used to solve systems with more equations and unknowns. As a motivational example, consider the the following problem.

1. Mathematical Formulation of a Practical Problem

Two large tanks, each holding 24 liters of brine, are interconnected by two pipes. Fresh water flows into tank A a the rate of 6 L/min, and fluid is drained out tank B at the same rate. Also, 8 L/min of fluid are pumped from tank A to tank B and 2 L/min from tank B to tank A. The solutions in each tank are well stirred sot that they are homogeneous. If, initially, tank A contains 5 in solution and Tank B contains 2 kg, find the mass of salt in the tanks at any time t.

To solve this problem, let $x(t)$ and $y(t)$ be the mass of salt in tanks A and B respectively. The variables x, y satisfy the system

$$\begin{aligned} \frac{dx}{dt} &= \tfrac{-1}{3}x + \tfrac{1}{12}y, \\ \frac{dy}{dt} &= \tfrac{1}{3}x - \tfrac{1}{3}y. \end{aligned} \tag{6.1}$$

The first equation gives $y = 12\frac{dx}{dt} + 4x$. Substituting this in the second equation and simplifying, we get

$$\frac{d^2x}{dt^2} + \frac{2}{3}\frac{dx}{dt} + \frac{1}{12}x = 0.$$

The general solution of this DE is

$$x = c_1 e^{-t/2} + c_2 e^{-t/6}.$$

This gives $y = 12\frac{dx}{dt} + 4x = -2c_1 e^{-t/2} + 2c_2 e^{-t/6}$. Thus the general solution of the system is

$$\begin{aligned} x &= c_1 e^{-t/2} + c_2 e^{-t/6}, \\ y &= -2c_1 e^{-t/2} + 2c_2 e^{-t/6}. \end{aligned} \tag{6.2}$$

These equations can be written in matrix form as

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-t/2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-t/6} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Using the initial condition $x(0) = 5$, $y(0) = 2$, we find $c_1 = 2$, $c_2 = 3$. Geometrically, these equations are the parametric equations of a curve (trajectory of the DE) in the xy-plane (phase plane of the DE). As $t \to \infty$ we have $(x(t), y(t)) \to (0, 0)$. The constant solution $x(t) = y(t) = 0$ is called an **equilibrium solution** of our system. This solution is said to be **asymptotically stable** if the general solution converges to it as $t \to \infty$. A system is called **stable** if the trajectories are all bounded as $t \to \infty$.

Our system can be written in matrix form as $\frac{dX}{dt} = AX$ where

$$A = \begin{pmatrix} -1/3 & 1/12 \\ 1/3 & -1/3 \end{pmatrix} X.$$

The 2×2 matrix A is called the matrix of the system. The polynomial

$$r^2 - \mathrm{tr}(A)r + \det(A) = r^2 + \frac{2}{3}r + \frac{1}{12}$$

where $\mathrm{tr}(A)$ is the trace of A (sum of diagonal entries) and $\det(A)$ is the determinant of A is called the **characteristic polynomial** of A. Notice that this polynomial is the characteristic polynomial of the differential equation for x. The equations

$$A \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{-1}{2} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{-1}{6} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

identify $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as eigenvectors of A with eigenvalues $-1/2$ and $-1/6$ respectively. More generally, a non-zero column vector X is an **eigenvector** of a square matrix A with **eigenvalue** r if $AX = rX$ or , equivalently, $(rI - A)X = 0$. The latter is a homogeneous system of linear equations with coefficient matrix $rI - A$. Such a system has a non-zero solution if and only if $\det(rI - A) = 0$. Notice that

$$\det(rI - A) = r^2 - (a+d)r + ad - bc$$

is the characteristic polynomial of A.

If, in the above mixing problem, brine at a concentration of 1/2 kg/L was pumped into tank A instead of pure water the system would be

$$\begin{aligned} \frac{dx}{dt} &= \tfrac{-1}{3}x + \tfrac{1}{12}y + 3, \\ \frac{dy}{dt} &= \tfrac{1}{3}x - \tfrac{1}{3}y, \end{aligned} \tag{6.3}$$

a non-homogeneous system. Here an equilibrium solution would be $x(t) = a, y(t) = b$ where (a, b) was a solution of

$$\begin{aligned} \tfrac{-1}{3}x + \tfrac{1}{12}y &= -3, \\ \tfrac{1}{3}x - \tfrac{1}{3}y &= 0. \end{aligned} \tag{6.4}$$

In this case $a = b = 12$. The variables $x^* = x - 12, y^* = y - 12$ then satisfy the homogeneous system

$$\begin{aligned} \frac{dx^*}{dt} &= \tfrac{-1}{3}x^* + \tfrac{1}{12}y^*, \\ \frac{dy^*}{dt} &= \tfrac{1}{3}x^* - \tfrac{1}{3}y^*. \end{aligned} \tag{6.5}$$

Solving this system as above for x^*, y^* we get $x = x^* + 12, y = y^* + 12$ as the general solution for x, y.

2. (2 × 2) System of Linear Equations

We now describe the solution of the system $\frac{dX}{dt} = AX$ for an arbitrary 2×2 matrix A. In practice, one can use the elimination method or the eigenvector method but we shall use the eigenvector method as it gives an explicit description of the solution. There are three main cases depending on whether the discriminant

$$\Delta = \operatorname{tr}(A)^2 - 4\det(A)$$

of the characteristic polynomial of A is > 0, < 0, $= 0$.

2.1 Case 1: $\Delta > 0$

In this case the roots r_1, r_2 of the characteristic polynomial are real and unequal, say $r_1 < r_2$. Let P_i be an eigenvector with eigenvalue r_i.

Then P_1 is not a scalar multiple of P_2 and so the matrix P with columns P_1, P_2 is invertible. After possibly replacing P_2 by $-P_2$, we can assume that $\det(P) > 0$. The equation

$$AP = P\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

shows that

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

If we make the change of variable $X = PU$ with $U = \begin{pmatrix} u \\ v \end{pmatrix}$, our system becomes

$$P\frac{dU}{dt} = APU \quad \text{or} \quad \frac{dU}{dt} = P^{-1}APU.$$

Hence, our system reduces to the uncoupled system

$$\frac{du}{dt} = r_1 u, \quad \frac{dv}{dt} = r_2 v$$

which has the general solution $u = c_1 e^{r_1 t}$, $v = c_2 e^{r_2 t}$. Thus the general solution of the given system is

$$X = PU = uP_1 + vP_2 = c_1 e^{r_1 t} P_1 + c_2 e^{r_2 t} P_2.$$

Since $\mathrm{tr}(A) = r_1 + r_2$, $\det(A) = r_1 r_2$, we see that $x(t), y(t) = (0, 0)$ is an asymptotically stable equilibrium solution if and only if $\mathrm{tr}(A) < 0$ and $\det(A) > 0$. The system is unstable if $\det(A) < 0$ or $\det(A) \geq 0$ and $\mathrm{tr}(A) \geq 0$.

2.2 Case 2: $\Delta < 0$

In this case the roots of the characteristic polynomial are complex numbers

$$r = \alpha \pm i\omega = \mathrm{tr}(A)/2 \pm i\sqrt{\Delta/4}.$$

The corresponding eigenvectors of A are (complex) scalar multiples of

$$\begin{pmatrix} 1 \\ \sigma \pm i\tau \end{pmatrix}$$

where $\sigma = (\alpha - a)/b$, $\tau = \omega/b$. If X is a real solution we must have $X = V + \overline{V}$ with

$$V = \frac{1}{2}(c_1 + ic_2)e^{\alpha t}(\cos(\omega t) + i\sin(\omega t))\begin{pmatrix} 1 \\ \sigma + i\tau \end{pmatrix}.$$

It follows that

$$X = e^{\alpha t}(c_1 \cos(\omega t) - c_2 \sin(\omega t)) \begin{pmatrix} 1 \\ \sigma \end{pmatrix} + e^{\alpha t}(c_1 \sin(\omega t) + c_2 \cos(\omega t)) \begin{pmatrix} 0 \\ \tau \end{pmatrix}.$$

The trajectories are spirals if $\mathrm{tr}(A) \neq 0$ and ellipses if $\mathrm{tr}(A) = 0$. The system is asymptotically stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) > 0$.

2.3 Case 3: $\Delta = 0$

Here the characteristic polynomial has only one root r. If $A = rI$ the system is

$$\frac{dx}{dt} = rx, \quad \frac{dy}{dt} = ry.$$

which has the general solution $x = c_1 e^{rt}$, $y = c_2 e^{rt}$. Thus the system is asymptotically stable if $\mathrm{tr}(A) < 0$, stable if $\mathrm{tr}(A) = 0$ and unstable if $\mathrm{tr}(A) > 0$.

Now suppose $A \neq rI$. If P_1 is an eigenvector with eigenvalue r and P_2 is chosen with $(A - rI)P_1 \neq 0$, the matrix P with columns P_1, P_2 is invertible and

$$P^{-1}AP = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}.$$

Setting as before $X = PU$ we get the system

$$\frac{du}{dt} = ru + v, \quad \frac{dv}{dt} = rv$$

which has the general solution $u = c_1 e^{rt} + c_2 t e^{rt}$, $v = c_2 e^{rt}$. Hence the given system has the general solution

$$X = uP_1 + vP_2 = (c_1 e^{rt} + c_2 t e^{rt})P_1 + c_2 e^{rt} P_2.$$

The trajectories are asymptotically stable if $\mathrm{tr}(A) < 0$ and unstable if $\mathrm{tr}(A) \geq 0$.

A non-homogeneous system $\frac{dX}{dt} = AX + B$ having an equilibrium solution $x(t) = x_1, y(t) = y_1$ can be solved by introducing new variables $x^* = x - x_1, y^* = y - y_1$. Since $AX^* + B = 0$ we have

$$\frac{dX^*}{dt} = AX^*,$$

a homogeneous system which can be solved as above.

(*) PART (II): EIGENVECTOR METHOD

In this lecture we will apply the eigenvector method to the solution of a second order system of the type arising in the solution of a mass-spring system with two masses. The system we will consider consists of two masses with mass m_1, m_2 connected by a spring with spring constant k_2. The first mass is attached to the ceiling of a room by a spring with spring constant k_1 and the second mass is attached to the floor by a spring with spring constant k_3 at a point immediately below the point of attachment to the ceiling. Assume that the system is under tension and in equilibrium. If $x_1(t)$, $x_1(t)$ are the displacements of the two masses from their equilibrium position at time t, the positive direction being upward, then the motion of the system is determined by the system

$$\begin{aligned} m_1\frac{d^2x_1}{dt^2} &= -k_1x_1 - k_2(x_1 - x_2) = -(k_1 + k_2)x_1 + k_2x_2, \\ m_2\frac{d^2x_2}{dt^2} &= k_2(x_1 - x_2) - k_3x_2 = k_2x_1 - (k_2 + k_3)x_2. \end{aligned} \tag{6.6}$$

The system can be written in matrix form $\frac{d^2X}{dt^2} = AX$ where

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad A = \begin{pmatrix} -(k_1+k_2)/m_1 & k_2/m_2 \\ k_2/m_1 & -(k_2+k_3)/m_2 \end{pmatrix}.$$

The characteristic polynomial of A is

$$r^2 + \frac{m_2(k_1+k_2) + m_1(k_2+k_3)}{m_1m_2}r + \left[\frac{(k_1+k_2)(k_2+k_3)}{m_1m_2} - \frac{k_2^2}{m_1m_2}\right].$$

The discriminant of this polynomial is

$$\begin{aligned}\Delta = \frac{1}{m_1^2 m_2^2}\Big\{\Big[m_2(k_1+k_2)+m_1(k_2+k_3)\Big]^2 \\ -4(k_1+k_2)(k_2+k_3)m_1m_2 + 4k_2^2 m_1 m_2\Big\} \\ = \frac{(m_2(k_1+k_2)-m_1(k_2+k_3))^2 + 4m_1m_2{k_2}^2}{m_1^2m_2^2} > 0.\end{aligned} \tag{6.7}$$

Hence the eigenvalues of A are real, distinct and negative since the trace of A is negative while the determinant is positive. Let $r_1 > r_2$ be the eigenvalues of A and let

$$P_1 = \begin{pmatrix} 1 \\ s_1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ s_2 \end{pmatrix}$$

be (normalized) eigenvectors with eigenvalues r_1, r_2 respectively. We have

$$s_1 = \frac{m_1 r_1 + k_1 + k_2}{k_2}, \quad s_2 = \frac{m_1 r_2 + k_1 + k_2}{k_2}$$

and, if P is the matrix with columns P_1, P_2, we have

$$P^{-1}AP = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}.$$

If we make a change of variables $X = PY$ with $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, we have

$$\frac{d^2Y}{dt^2} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

so that our system in the new variables y_1, y_2 is

$$\begin{aligned}\tfrac{d^2y_1}{dt^2} &= r_1 y_1 \\ \tfrac{d^2y_2}{dT^2} &= r_2 y_2.\end{aligned} \tag{6.8}$$

Setting $r_i = -\omega_i^2$ with $\omega_i > 0$, this uncoupled system has the general solution

$$y_1 = A_1\sin(\omega_1 t) + B_1\cos(\omega_1 t), \quad y_2 = A_2\sin(\omega_2 t) + B_2\cos(\omega_2 t).$$

Since $X = PY = y_1P_1 + y_2P_2$, we obtain the general solution

$$X = (A_1\sin(\omega_1 t) + B_1\cos(\omega_1 t))P_1 + (A_2\sin(\omega_2 t) + B_2\cos(\omega_2 t))P_2.$$

The two solutions with $Y(0) = P_i$ are of the form

$$X = (A\sin(\omega_i t) + B\cos(\omega_i t))P_i = \sqrt{A^2+B^2}\sin(\omega_i t + \theta_i)P_i.$$

These motions are simple harmonic with frequencies $\omega_i/2\pi$ and are called the **fundamental motions** of the system. Since any motion of the system is the sum (superposition) of two such motions any periodic motion of the system must have a period which is an integer multiple of both the fundamental periods $2\pi/\omega_1$, $2\pi/\omega_2$. This happens if and only if ω_1/ω_2 is a rational number. If $X'(0) = 0$, the fundamental motions are of the form

$$X = B_i \cos(\omega_i t) P_i$$

and if $X(0) = 0$ they are of the form

$$X = A_i \sin(\omega_i t) P_i.$$

These four motions are a basis for the solution space of the given system. The motion is completely determined once $X(0)$ and $X'(0)$ are known since

$$X(0) = PY(0) = P\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad X'(0) = PY'(0) = P\begin{pmatrix} \omega_1 A_1 \\ \omega_2 A_2 \end{pmatrix}.$$

As a particular example, consider the case where $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. The system is symmetric and

$$A = \frac{k}{m}\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

a symmetric matrix. The characteristic polynomial is

$$r^2 + 4\frac{k}{m}r + 3\frac{k^2}{m^2} = (r + \frac{k}{m})(r + 3\frac{k}{m}).$$

The eigenvalues are $r_1 = -k/m$, $r_2 = -3k/m$. The fundamental frequencies are $\omega_1 = \sqrt{k/m}$, $\omega_2 = \sqrt{3k/m}$. The normalized eigen-vectors are

$$P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The fundamental motions with $X'(0) = 0$ are

$$X = A\cos(\sqrt{k/m}\ t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X = A\cos(\sqrt{3k/m}\ t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since the ratio of the fundamental frequencies is $\sqrt{3}$, an irrational number, theses are the only two periodic motions of the mass-spring system where the masses are displaced and then let go.

Odds and Ends

If $y = f(x)$ is a solution of the autonomous DE $y^n = f(y, y', \ldots, y^{n-1})$ then so is $y = f(x + a)$ for any real number a. If the DE is linear and homogeneous with fundamental set $y_1, y_2, \ldots, y_n$ then we must have identities of the form

$$y_1(x + a) = c_2 y_2 + c_3 y_3 + \cdots + c_n y_n.$$

For example, consider the DE $y'' + y = 0$. Here $\sin(x), \cos(x)$ is a fundamental set so we must have an identity of the form

$$\sin(x + a) = A \sin(x) + B \cos(x).$$

Differentiating, we get $\cos(x + a) = A \cos(x) - B \sin(x)$. Setting $x = 0$ in these two equations we find $A = \cos(a)$, $B = \sin(a)$. We obtain in this way the addition formulas for the sine and cosine functions:

$$\sin(x + a) = \sin(x)\cos(a) + \sin(a)\cos(x),$$
$$\cos(x + a) = \cos(x)\cos(a) - \sin(x)\sin(a).$$

The numerical methods for solving DE's can be extended to systems virtually without change. In this way we can get approximate solutions for higher order DE's. For more details consult the text (Chapter 5).

Appendix A
ASSIGNMENTS AND SOLUTIONS

Assignment 2B: due Thursday, September 21, 2000

1 Find the solution of the initial value problem

$$yy' = x(y^2 - 1)^{4/3}, \quad y(0) = b > 0.$$

What is its interval of definition? (Your answer will depend on the value of b.) Sketch the graph of the solution when $b = 1/2$ and when $b = 2$

2 Find the general solution of the differential equation

$$\frac{dy}{dx} = y + e^{2x}y^3.$$

3 Solve the initial value problem

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x}, \quad y(1) = -4.$$

4 Solve the initial value problem

$$(e^x - 1)\frac{dy}{dx} + ye^x + 1 = 0, \quad y(1) = 1.$$

Solutions for Assignment 2B

1 Separating variables and integrating we get

$$\int \frac{yy'dx}{(y^2-1)^{4/3}} = \frac{x^2}{2} + C_1$$

from which, on making the change of variables $u = y^2$, we get

$$\frac{1}{2}\int (u-1)^{-4/3}du = \frac{x^2}{2} + C_1.$$

Integrating and simplifying, we get

$$(u-1)^{-1/3} = C - x^2/3 \quad \text{with} \quad C = -2C_1/3.$$

Hence $(y^2 - 1)^{-1/3} = C - x^2/3$. Then $y(0) = b$ gives $C = (b^2 - 1)^{-1/3}$. Since $b > 0$ we must have

$$y = \sqrt{1 + \frac{1}{(C - x^2)^3}}.$$

If $b < 1$ then y is defined for all x while, if $b > 1$, the solution y is defined only for $|x| < \sqrt{3C}$.

2 This is a Bernoulli equation. To solve it, divide both sides by y^3 and make the change of variables $u = 1/y^2$. This gives

$$u' = -2u - 2e^{2x}$$

after multiplication by -2. We now have a linear equation whose general solution is

$$u = -e^{2x}/2 + Ce^{-2x}.$$

This gives a 1-paramenter family of solutions

$$y = \frac{\pm 1}{\sqrt{Ce^{-2x} - e^{2x}/2}} = \frac{\pm e^x}{\sqrt{C - e^{4x}/2}}$$

of the original DE. Given (x_0, y_0) with $y_0 \neq 0$ there is a unique value of C such that the solution satisfies $y(x_0) = y_0$. It is not the general solution as it omits the solution $y = 0$. Thus the general solution is comprised of the functions

$$y = 0, \quad y = \frac{\pm e^x}{\sqrt{C - e^{4x}/2}}.$$

3 This is a homogeneous equation. Setting $u = y/x$, we get

$$xu' + u = 1/u + u.$$

This gives $xu' = 1/u$, a separable equation from which we get $uu' = 1/x$. Integrating, we get

$$u^2/2 = \ln|x| + C_1$$

and hence $y^2 = x^2 \ln(x^2) + Cx^2$ with $C = 2C_1$. For $y(1) = -4$ we must have $C = 16$ and

$$y = -x\sqrt{\ln(x^2) + 16}, \quad x > 0.$$

4 This is a linear equation which is also exact. The general solution is $F(x, y) = C$ where

$$\frac{\partial F}{\partial x} = ye^x - 1, \quad \frac{\partial F}{\partial y} = e^y - 1.$$

Integrating the first equation partially with respect to x we get

$$F(x, y) = ye^x + x + \phi(y)$$

from which $\frac{\partial F}{\partial y} = e^x + \phi'(y) = e^y - 1$ which gives $\phi(y) = -y$ (up to a constant) and hence

$$F(x, y) = ye^x + x - y = C.$$

For $y(1) = 1$ we must have $C = e$ and so the solution is

$$y = \frac{e - x}{e^x - 1}, \quad (x > 0).$$

Assignment 3B: due Thursday, September 28, 2000

1 One morning it began to snow very hard and continued to snow steadily through the day. A snowplow set out at 8:00 A.M. to clear a road, clearing 2 miles by 11:00 A.M. and an additional mile by 1:00 P.M. At what time did it start snowing. (You may assume that it was snowing at a constant rate and that the rate at which the snowplow could clear the road was inversely proportional to the depth of the snow.)

2 Find, in implicit form, the general solution of the differential equation

$$y^3 + 4ye^x + (2e^x + 3y^2)y' = 0.$$

Given x_0, y_0, is it always possible to find a solution such that $y(x_0) = y_0$? If so, is this solution unique? Justify your answers.

Solution for Assignment 3B

1 Let x be the distance travelled by the snow plow in t hours with $t = 0$ at 8 AM. Then if it started snowing at $t = -b$ we have

$$\frac{dx}{dt} = \frac{a}{t+b}.$$

The solution of this DE is $x = a\ln(t+b) + c$. Since $x(0) = 0$, $x(3) = 2$, $x(5) = 3$, we have $a\ln b + c = 0$, $a\ln(3+b) = 2$, $a\ln(5+b) + c = 3$ from which

$$a\ln\frac{3+b}{b} = 2, \quad a\ln\frac{5+b}{3+b} = 1.$$

Hence $(3+b)/b = (5+b)^2/(3+b)^2$ from which $b^2 - 2b - 27 = 0$. The positive root of this equation is $b = 1 + 2\sqrt{7} \approx 6.29$ hours. Hence it started snowing at $1:42:36$ AM.

2 The DE $y^3 + 4ye^x + (2e^x + 3y^2)y' = 0$ has an integrating factor $\mu = e^x$. The solution in implicit form is $2e^{2x}y + y^3e^x = C$. There is a unique solution with $y(x_0) = y_0$ for any x_0, y_0 by the fundamental existence and uniqueness theorem since the coefficient of y' in the DE is never zero and hence

$$f(x,y) = \frac{-y^3 - 4ye^x}{2e^x + 3y^2}$$

and its partial derivative f_y are continuously differentiable on $\mathcal{R}^2$.

Alternately, since the partial derivative of $y^3e^x + 2ye^{2x}$ with respect to y is never zero, the implicit function theorem guarantees the existence of a unique function $y = y(x)$, with $y(x_0) = y_0$ and defined in some neighborhood of x_0, which satisfies the given DE.

Assignment 4B: due Tuesday, October 24, 2000

1 (a) Show that the differential equation $M + Ny' = 0$ has an integrating factor which is a function of $z = x + y$ only if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N}$$

is a function of z only.

(b) Use this to solve the differential equation

$$x^2 + 2xy - y^2 + (y^2 + 2xy - x^2)y' = 0.$$

2 Solve the differential equations

(a) $xy'' = y' + x, \quad (x > 0);$

(b) $y(y-1)y'' + {y'}^2 = 0.$

3 Solve the differential equations

(a) $y''' - 3y' + 2y = e^x;$

(b) $y^{(iv)} - 2y''' + 5y'' - 8y' + 4y = \sin(x).$

4 Show that the functions $\sin(x),\ \sin(2x),\ \sin(3x)$ are linearly independent. Find a homogeneous linear ODE having these functions as part of a basis for its solution space. Show that it is not possible to find such an ODE with these functions as a basis for its solution space.

Solutions to Assignment 4B

1 (a) Suppose that $M + Ny' = 0$ has an integrating factor u which is a function of $z = x + y$. Then $\frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{\partial x}$ gives

$$u(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = N\frac{\partial u}{\partial x} - N\frac{\partial u}{\partial y}.$$

By the chain rule we have

$$\frac{\partial u}{\partial x} = \frac{du}{dz}\frac{\partial z}{\partial x} = \frac{du}{dz}, \quad \frac{\partial u}{\partial y} = \frac{du}{dz}\frac{\partial z}{\partial y} = \frac{du}{dz},$$

so that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = \frac{-1}{u}\frac{du}{dz},$$

which is a function of z. Conversely, suppose that

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = f(z),$$

with $z = x + y$. Now define $u = u(z)$ to be a solution of the linear DE $\frac{du}{dz} = -f(z)u$. Then

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = \frac{-1}{u}\frac{du}{dz},$$

which is equivalent to $\frac{\partial(uM)}{\partial y} = \frac{\partial(uN)}{\partial x}$, i.e., that u is an integrating factor of $M + Ny'$ which is a function of $z = x + y$ only.

(b) For the DE $x^2 + 2xy - y^2 + (y^2 + 2xy - x^2)y' = 0$ we have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M - N} = \frac{2}{x + y} = \frac{2}{z}.$$

If we define

$$u = e^{\int -2dz/z} = e^{-2\ln z} = 1/z^2 = 1/(x + y)^2$$

then u is an integrating factor so that there is a function $F(x, y)$ with

$$\frac{\partial F}{\partial x} = uM = \frac{x^2 + 2xy - y^2}{(x + y)^2}, \quad \frac{\partial F}{\partial y} = uN = \frac{y^2 + 2xy - x^2}{(x + y)^2}.$$

Integrating the first DE partially with respect to x, we get

$$F(x, y) = \int (1 - \frac{2y^2}{(x + y)^2}dx = x + \frac{2y^2}{x + y} + \phi(y).$$

Differentiating this with respect to y and using the second DE, we get

$$\frac{y^2 + 2xy - x^2}{(x + y)^2} = \frac{\partial F}{\partial y} = \frac{2y^2 + 4xy}{(x + y)^2} + \phi'(y)$$

so that $\phi'(y) = -1$ and hence $\phi(y) = -y$ (up to a constant). Thus

$$F(x,y) = x + \frac{2y^2}{x+y} - y = \frac{x^2+y^2}{x+y}.$$

Thus the general solution of the DE is $F(x,y) = C$ or $x + y = 0$ which is the only solution that was missed by the integrating factor method. The first solution is the family of circles $x^2 + y^2 - Cx - Cy = 0$ passing through the origin and center on the line $y = x$. Through any point $\neq (0,0)$ there passes a unique solution.

2 (a) The dependent variable y is missing from the DE $xy'' = y' + x$. Set $w = y'$ so that $w' = y''$. The DE becomes $xw' = w + x$ which is a linear DE with general solution $w = x\ln(x) + C_1x$. Thus $y' = x\ln(x) + C_1$ which gives

$$y = \frac{x^2}{2}\ln(x) - \frac{x^2}{4} + C_1\frac{x^2}{2} + C_2 = \frac{x^2}{2}\ln(x) + A\frac{x^2}{2} + B$$

with A, B arbitrary constants.

(b) The independent variable x is missing DE $y(y-1) + {y'}^2 = 0$. Note that $y = C$ is a solution. We assume that $y \neq C$. Let $w = y'$. Then

$$y'' = \frac{dw}{dx} = \frac{dw}{dy}\frac{dy}{dx} = w\frac{dw}{dy}$$

so that the given DE becomes $y(y-1)\frac{dw}{dy} = -w$ after dividing by w which is not zero. Separating variables and integrating, we get

$$\int \frac{dw}{w} = -\int \frac{dy}{y(y-1)}$$

which gives $\ln|w| = \ln|y| - \ln|y-1| + C_1$. Taking exponentials, we get

$$w = \frac{Ay}{y-1}.$$

Since $w = y'$ we have a separable equation for y. Separating variables and integrating, we get $y - \ln|y| = Ax + B_1$. Taking exponentials, we get $e^y/y = Be^{Ax}$ with A arbitrary and $B \neq 0$ as an implicit definition of the non-constant solutions.

3 (a) The associated homogeneous DE is $(D^3 - 3Dy + 2)(y) = 0$. Since

$$D^3 - 3D + 2 = (D-1)^2(D-2)$$

this DE has the general solution $y_h = (A + Bx)e^x + Ce^{2x}$. Since the RHS of the original DE is killed by $D - 1$, a particular solution y_p of it satisfies the DE

$$(D-1)^3(D-2) = 0$$

and so must be of the form $(A + Bx + Ex^2)e^x + Ce^{2x}$. Since we can delete the terms which are solutions of the homogeneous DE, we can take $y_p = Ex^2e^x$. Substituting this in the original DE, we find $E = 1/6$ so that the general solution is

$$y = y_h + y_p = (A + Bx)E^x + Ce^{2x} + x^2e^x/6.$$

(b) The associated homogeneous DE is $(D^4 - 2D^3 + 5D^2 - 8D + 4)(y) = 0$. Since

$$D^4 - 2D^3 + 5D^2 - 8D + 4 = (D-1)^2(D+4)$$

this DE has general solution $y_h = (A + Bx)e^x + E\sin(2x) + F\cos(2x)$. A particular solution y_p is a solution of the DE

$$(D^2+1)(D-1)^2(D^2+4)(y) = 0$$

so that there is a particular solution of the form $C_1\cos(x) + C_2\sin(x)$. Substituting in the original equation, we find $C_1 = 1/6$, $C_2 = 0$. Hence

$$y = y_h + y_p = (A + Bx)e^x + E\sin(2x) + F\cos(2x) + \frac{1}{6}\cos(x)$$

is the general solution.

4 (a)

$$W(\sin(x), \sin(2x), \sin(3x)) = \begin{pmatrix} \sin(x) & \sin(2x) & \sin(3x) \\ \cos(x) & 2\cos(2x) & 3\cos(3x) \\ -\sin(x) & -4\sin(2x) & -9\sin(3x) \end{pmatrix}$$

so that $W(\pi/2) = -16 \neq 0$. Hence $\sin(x), \sin(2x), \sin(3x)$ are linearly independent.

(b) The DE $(D^2+1)(D^2+4)(D^2+9)(y) = 0$ has basis

$$\sin(x), \sin(2x)\sin(3x)\cos(x), \cos(2x)\cos(3x)$$

and the given functions are part of it.

(c) Since the Wronskian of the given functions is zero at $x = 0$ it cannot be a fundamental set for a necessarily third order linear DE.

Assignment 5B: due Thursday, Oct. 24, 2002

1 Find the general solution of the differential equation

$$y'' + 4y' + 4y = e^{-2x}\ln(x), \quad (x > 0).$$

2 Given that $y_1 = \cos(x)/\sqrt{x}$, $y_2 = \sin(x)/\sqrt{x}$ are linearly independent solutions of the differential equation

$$x^2y'' + xy' + (x^2 - 1/4)y = 0, \quad (x > 0),$$

find the general solution of the equation

$$x^2y'' + xy' + (x^2 - 1/4)y = x^{5/2}, \quad (x > 0).$$

3 Find the general solution of the equation

$$x^2y'' + 3xy' + y = 1/x\ln(x), \quad (x > 0).$$

4 Find the general solution of the equation

$$(1 - x^2)y'' - 2xy' + 2y = 0, \quad (-1 < x < 1)$$

given that $y = x$ is a solution.

5 Find the general solution of the equation

$$xy'' + xy' + y = x, \quad (x > 0).$$

Assignment 5 Solutions

1 The differential equation in operator form is $(D+2)^2(y) = e^{-2x}\ln(x)$. Multiplying both sides by e^{2x} and using the fact that $D+2 = e^{-2x}De^{2x}$, we get $D^2(e^{2x}y) = \ln(x)$. Hence

$$e^{2x}y = \frac{1}{2}x^2\ln x - \frac{3}{4}x^2 + Ax + B$$

from which we get $y = Axe^{2x} + Be^{2x} + \frac{1}{2}x^2e^{2x}\ln x - \frac{3}{4}x^2e^{-2x}$. Variation of parameters could also have been used but the solution would have been longer.

2 The given functions $y1 = \cos x/\sqrt{x}$, $y_2 = \sin x/\sqrt{x}$ are linearly independent and hence a fundamental set of solutions for the DE

$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = 0.$$

We only have to find a particular y solution of the normalized DE

$$y'' + \frac{1}{x}y' + (1 - \frac{1}{4x^2})y = \sqrt{x}.$$

Using variation of parameters there is a solution of the form $y = uy_1 + vy_2$ with

$$\begin{aligned} u'y_1 + v'y_2 &= 0, \\ u'y_1' + v'y_2' &= \sqrt{x}. \end{aligned} \tag{A.1}$$

By Crammer's Rule we have

$$u' = \frac{\begin{vmatrix} 0 & \frac{\sin x}{\sqrt{x}} \\ \sqrt{x} & \frac{-\sin x + 2x\sin x}{2x\sqrt{x}} \end{vmatrix}}{\begin{vmatrix} \frac{\cos x}{\sqrt{x}} & \frac{\sin x}{\sqrt{x}} \\ \frac{-\cos x - 2x\sin x}{2x\sqrt{x}} & \frac{-\sin x + 2x\sin x}{2x\sqrt{x}} \end{vmatrix}} = -x\sin x$$

$$v' = \frac{\begin{vmatrix} \frac{\cos x}{\sqrt{x}} & 0 \\ \frac{-\cos x - 2x\sin x}{2x\sqrt{x}} & \sqrt{x} \end{vmatrix}}{\begin{vmatrix} \frac{\cos x}{\sqrt{x}} & \frac{\sin x}{\sqrt{x}} \\ \frac{-\cos x - 2x\sin x}{2x\sqrt{x}} & \frac{-\sin x + 2x\sin x}{2x\sqrt{x}} \end{vmatrix}} = x\cos x$$

so that $u = x\cos x - \sin x$, $v = x\sin x + \cos x$ and

$$y = (x\cos x - \sin x)\frac{\cos x}{\sqrt{x}} + (x\sin x + \cos x)\frac{\sin x}{\sqrt{x}} = \sqrt{x}.$$

Hence the general solution of the DE $x^2y'' + xy' + (x^2 - 1/4)y = x^{5/2}$ is

$$y = A\frac{\cos x}{\sqrt{x}} + B\frac{\sin x}{\sqrt{x}} + \sqrt{x}.$$

3 This is an Euler equation. So we make the change of variable $x = e^t$. The given DE becomes

$$(D(D-1) + 3D + 1)(y) = e^{-t}/t$$

where $D = \frac{d}{dt}$. Since $D(D-1)+3D+1 = D^2+2D+1 = (D+1)^2$, the given DE is

$$(D+1)^2(y) = e^{-t}/t.$$

Multiplying both sides by e^t and using $e^t(D+1) = De^t$, we get

$$D^2(e^t y) = 1/t$$

from which $e^t y = t\ln t + At + B$ and

$$y = Ate^{-t} + Be^{-t} + te^{-t}\ln t = A\frac{\ln x}{x} + \frac{B}{x} + \frac{\ln x}{x}\ln(\ln(x)),$$

the general solution of the given DE.

4 Using reduction of order, we look for a solution of the form $y = xv$. Then $y' = xv' + v$, $y'' = xv'' + 2v'$ and

$$(1-x^2)(xv''+2v') - 2x(xv'+v) + 2xv = 0$$

which simplifies to

$$v'' + \frac{2-4x^2}{x-x^3}v' = 0$$

which is a linear DE for v'. Hence

$$v' = e^{\int \frac{2-4x^2}{x^3-x}dx}.$$

Since

$$\frac{2-4x^2}{x^3-x} = \frac{-2}{x} + \frac{-1}{x+1} + \frac{1}{1-x},$$

we have

$$v' = 1/x^2(1-x)(x+1) = -\frac{1}{x} + \frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1}.$$

Hence $v = -\frac{1}{x} + \frac{1}{2}\ln\frac{1+x}{1-x}$ and the general solution of the given DE is

$$y = Ax + B(-1 + \frac{x}{2}\ln\frac{1+x}{1-x}.$$

5 This DE is exact and can be written in the form

$$\frac{d}{dx}(xy' + (x-1)y) = x$$

so that $xy' + (x-1)y = x^2/2 + C$. This is a linear DE. Normalizing, we get

$$y' + (1-1/x)y = x/2 + C/x.$$

An integrating factor for this equation is e^x/x.

$$\frac{d}{dx}(\frac{e^x}{x}y) = e^x/2 + Ce^x/x^2,$$

$$\frac{e^x}{x}y = e^x/2 + C\int\frac{e^x}{x^2}dx + D,$$

$$y = x/2 + Cxe^{-x}\int\frac{e^x}{x^2}dx + Dxe^{-x}.$$

Assignment 7B: due Thursday, November 21, 2000

For each of the following differential equations show that $x = 0$ is a regular singular point. Also, find the indicial equation and the general solution using the Frobenius method.

1 $9x^2y'' + 9xy' + (9x - 1)y = 0.$

2 $xy'' + (1 - x)y' + y = 0.$

3 $x(x + 1)y'' + (x + 5)y' - 4y = 0.$

Solutions for Assignment 7(b)

1 The differential equation in normal form is

$$y'' + p(x)y' + q(x)y = y'' + \frac{1}{x}y' + \left(\frac{1}{x} - \frac{1}{9x^2}\right)y = 0$$

so that $x = 0$ is a singular point. This point is a regular singular point since

$$xp(x) = 1, \quad x^2q(x) = -\frac{1}{9} + x$$

are analytic at $x = 0$. The indicial equation is $r(r-1) + r - 1/9 = 0$ so that $r^2 - 1/9 = 0$, i.e., $r = \pm 1/3$. Using the method of Frobenius, we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Substituting this into the differential equation $x^2y'' + x^2p(x)y' + x^2q(x)y = 0$, we get

$$(r^2 - 1/9)a_0x^r + \sum_{n=1}^{\infty}(((n+r)^2 - 1/9)a_n + a_{n-1})x^{n+r} = 0.$$

In addition to $r = \pm 1/3$, we get the recursion equation

$$a_n = -\frac{a_{n-1}}{(n+r)^2 - 1/9} = -\frac{9a_{n-1}}{(3n+3r-1)(3n+3r+1)}$$

for $n \geq 1$. If $r = 1/3$, we have $a_n = -3a_{n-1}/n(3n+2)$ and

$$a_n = \frac{(-1)^n 3^n a_0}{n!5 \cdot 8 \cdots (3n+2)}.$$

Taking $a_0 = 1$, we get the solution

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n a_0}{n!5 \cdot 8 \cdots (3n+2)} x^n.$$

Similarly for $r = -1/3$, we get the solution

$$y_2 = x^{-1/3} \sum_{n=0}^{\infty} \frac{(-1)^n 3^n a_0}{n!1 \cdot 4 \cdots (3n-2)} x^n.$$

The general solution is $y = Ay_1 + By_2$.

2 The differential equation in normal form is

$$y'' + p(x)y' + q(x)y = y'' + (\frac{1}{x} - 1)y' + \frac{1}{x} = 0$$

so that $x = 0$ is a singular point. This point is a regular singular point since

$$xp(x) = 1 - x, \quad x^2q(x) = x$$

are analytic at $x = 0$. The indicial equation is $r(r-1) + r = 0$ so that $r^2 = 0$, i.e., $r = 0$. Using the method of Frobenius, we look for a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

Substituting this into the differential equation $x^2 y'' + x^2 p(x) y' + x^2 q(x) y = 0$, we get

$$r^2 a_0 x^r + \sum_{n=0}^{\infty} ((n+r)^2 a_n - (n+r-2) a_{n-1}) x^{n+r} = 0.$$

This yields the recursion equation

$$a_n = \frac{n+r-2}{(n+r)^2} a_{n-1}, \quad (n \geq 1).$$

Hence

$$a_n(r) = \frac{(r-1)r(r+1)\cdots(r+n-2)}{(r+1)^2(r+2)^2\cdots(r+n)^2} a_0.$$

Taking $r = 0$, $a_0 = 1$, we get the solution

$$y_1 = 1 - x.$$

To get a second solution we compute $a_n'(0)$. Using logarithmic differentiation, we get

$$a_n'(r) = a_n(r)\left(\frac{1}{r-1} + \frac{1}{r} + \cdots + \frac{1}{n+r-2} - \frac{2}{r+1} - \frac{2}{r+2} - \cdots - \frac{2}{r+n}\right).$$

Hence $a_1'(0) = 3a_0$ and $a_n'(r) = a_n(r)/r + a_n(r) b_n(r)$ for $n \geq 2$. Setting $r = 0$, we get for $n \geq 2$

$$a_n'(0) = \frac{(-1) \cdot 1 \cdot 2 \cdots \cdot (n-2)}{(n!)^2} a_0$$

from which $a_n = -(n-2)! a_0/(n!)^2$ for $n \geq 2$. Taking $a_0 = 1$, we get as second solution

$$y_2 = y_1 \ln(x) + 3x - \sum_{n=2}^{\infty} \frac{(n-2)!}{(n!)^2} x^n = y_1 \ln(x) + 4x - 1 + y_1.$$

The general solution is then $y = Ay_1 + B(y_1 \ln(x) + 4x - 1)$.

Assignment 8B: due Thursday, November 23, 2000

1 (a) Compute the Laplace transforms of the functions

$$t^2 \sin(t), \quad t^2 \cos(t).$$

(b) Find the inverse Laplace transforms of the functions

$$\frac{s}{(s^2+1)^3}, \quad \frac{1}{(s^2+1)^3}.$$

2 Using Laplace transforms, solve the initial value problem

$$y^{iv} - y = \sin(t), \quad y(0) = y'(0) = 1, y''(0) = y'''(0) = -1.$$

3 Using Laplace transforms, solve the system

$$\begin{aligned} \tfrac{dx}{dt} &= -2x + 3y, \\ \tfrac{dy}{dt} &= x - y \end{aligned} \qquad \text{(A.2)}$$

with the initial conditions $x(0) = 1, y(0) = -1$.

4 Using Laplace transforms, solve the initial value problem

$$y'' + 3y' + 2y = f(t), \quad y(0) = y'(0) = 0,$$

where

$$f(t) = \{ 1\ , \quad 0 \le t < 1, -1, \quad 1 \le t < \pi, \sin(t), \quad \pi \le t.$$

Solutions for Assignment 8(b)

1 (a)

$$\begin{aligned}
\mathcal{L}\{t\sin(t)\} &= -\frac{d}{ds}\frac{1}{s^2+1} = \frac{-2s}{(s^2+1)^2},\\
\mathcal{L}\{t\cos(t)\} &= -\frac{d}{ds}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}\\
\mathcal{L}\{t^2\sin(t)\} &= -\frac{d}{ds}\frac{-2s}{(s^2+1)^2} = \frac{6}{(s^2+1)^2} - \frac{8}{(s^2+1)^3},\\
\mathcal{L}\{t^2\cos(t)\} &= -\frac{8s}{(s^2+1)^3} + \frac{2s}{(s^2+1)^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathcal{L}^{-1}\{\frac{s}{(s^2+1)^3}\} &= -t^2\cos(t)/8 + t\sin(t)/8,\\
\mathcal{L}^{-1}\{\frac{1}{(s^2+1)^3}\} &= 3\sin(t)/8 - 3t\cos(t)/8 - t^2\sin(t)/8.
\end{aligned}$$

2 If $Y(s) = \mathcal{L}\{y(t)\}$, we have $(s^4-1)Y(s) - s^3 - s^2 + s + 1 = \frac{1}{s^2+1}$. Hence

$$\begin{aligned}
Y(s) &= \frac{1}{(s^4-1)(s^2+1)} + \frac{((s+1)(s^2-1)}{s^4-1}\\
&= \frac{1}{(s^4-1)(s^2+1)} + \frac{((s+1)}{s^2+1}\\
&= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{1}{s+1} + \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{2}\frac{1}{(s^2+1)^2} + \frac{s}{s^2+1}.
\end{aligned}$$

$$y(t) = e^t/8 - e^{-t}/8 + \sin(t)/2 + \cos(t) + t\cos(t)/4.$$

3 If $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$ we have

$$sX(s) - 1 = -2X(s) + 3Y(s), \quad sY(s) + 1 = X(s) - Y(s).$$

Hence we have

$$\begin{aligned}
X(s) &= \frac{(s-2)}{(s^2+3s-1)},\\
Y(s) &= \frac{(-1-s)}{(s^2+3s-1)}
\end{aligned}$$

and

$$\begin{aligned}
X(s) &= \left(\tfrac{7\sqrt{13}}{26} + 12\right)\tfrac{1}{s+(3+\sqrt{13})/2} + \left(\tfrac{-7\sqrt{13}}{26} + \tfrac{1}{2}\right)\tfrac{1}{s-(\sqrt{13}-3)/2},\\
Y(s) &= -\left(\tfrac{\sqrt{13}}{26} + 12\right)\tfrac{1}{s+(3+\sqrt{13})/2} + \left(\tfrac{\sqrt{13}}{26} + \tfrac{1}{2}\right)\tfrac{1}{s-(\sqrt{13}-3)/2},\\
x(t) &= \left(\tfrac{7\sqrt{13}}{26} + 12\right)e^{-(3+\sqrt{13})t/2} + \left(\tfrac{-7\sqrt{13}}{26} + \tfrac{1}{2}\right)e^{(\sqrt{13}-3)t/2},\\
y(t) &= -\left(\tfrac{\sqrt{13}}{26} + 12\right)e^{-(3+\sqrt{13})t/2} + \left(\tfrac{\sqrt{13}}{26} + \tfrac{1}{2}\right)e^{(\sqrt{13}-3)t/2}
\end{aligned}$$

4 We have $y'' + 3y' + 2y = 1 - 2u_1(t) + (\sin(t) + 1)u_\pi(t)$. If $Y(s) = \mathcal{L}\{y(t)\}$

$$(s^2 + 3s + 2)Y(s) = \frac{1}{2} - 2\frac{e^{-s}}{s} + e^{-\pi s}\left(\frac{-1}{s^2+1} + \frac{1}{s}\right),$$

since $\sin(t+\pi) = -\sin(t)$. Hence

$$\begin{aligned} Y(s) \;=\; & \frac{1}{s(s+1)(s+2)} + e^{-s}\frac{-2}{s(s+1)(s+2)} \\ & + e^{-\pi s}\left(\frac{-1}{(s^2+1)(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}\right). \end{aligned}$$

Since

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)},$$

$$\frac{1}{(s^2+1)(s+1)(s+2)} = \frac{1}{2(s+1)} - \frac{1}{5(s+2)} + \frac{1-3s}{10(s^2+1)},$$

we have

$$\begin{aligned} Y(s) = & \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)} + \left(\frac{-1}{s} + \frac{2}{s+1} - \frac{1}{s+2}\right)e^{-s} \\ & + \left(\frac{1}{2s} - \frac{3}{2(s+1)} + \frac{7}{10(s+2)} - \frac{1}{10(s^2+1)} - \frac{3s}{10(s^2+1)}\right)e^{-\pi s}. \end{aligned}$$

and

$$\begin{aligned} y(t) = & \frac{1}{2} - e^{-t} + \frac{e^{-2t}}{2} + \Big(-1 + 2e^{1-t} - e^{2-2t}\Big)u_1(t) \\ & + \left(\frac{1}{2} - \frac{3e^{\pi-t}}{2} + \frac{7e^{2\pi-2t}}{10} + \frac{\sin(t)}{10} - \frac{3\cos(t)}{10}\right)u_\pi(t). \end{aligned}$$

Hence

$$y(t) = \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, & 0 \le t < 1, \\ -\frac{1}{2} + (2e-1)e^{-t} + (\frac{1}{2} - e^2)e^{-2t}, & 1 \le t < \pi, \\ (2e - 1 - \frac{3}{2}e^{\pi})e^{-t} + (\frac{1}{2} - e^2 + \frac{7}{10}e^{2\pi})e^{-2t} & \\ \qquad + \frac{1}{10}\sin(t) - \frac{3}{10}\cos(t), & \pi \le t. \end{cases}$$